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ERROR PERFORMANCE OF DIRECT SEQUENCE SPREAD SPECTRUM SYSTEMS ON NONSELECTIVE GENERALIZED RAYLEIGH FADING CHANNELS

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University of Illinois at Urbana-Champaign, 1980

Abstract

The error performance of direct-sequence spread spectrum multiple-access communications systems on nonselective fading channels is investigated. The channels considered are those for which the fading intensity is slow, nonselective, and exhibit a generalized Rayleigh distribution. The generalized Rayleigh distribution encompasses the important special cases of Rician (specular plus Rayleigh) fading, Nakagami m-distributed fading, and Rayleigh fading. Results are obtained for the special cases of a single cochannel interferer without additive noise and multiple independent cochannel interferers with additive white Gaussian noise. The results are developed as a series expansion in the moments of the cross-correlation between the spreading codes (single interferer) or as a series expansion in the powers of the second moment of the code cross-correlation (multiple interferers). To develop simple estimates of the probability of error, empirical estimates of the low-order cross-correlation moments of the spreading codes (taken to be Gold codes) are developed from a generalized gamma density approximation based on the cross-correlation moments for both Gold codes and random codes. Numerical results for the probability of error are obtained with these estimates and compared with the results based on exact moments developed from the code cross-correlation probability density. (103 pages).

ERROR PERFORMANCE OF DIRECT SEQUENCE SPREAD
SPECTRUM SYSTEMS ON NONSELECTIVE
GENERALIZED RAYLEIGH FADING CHANNELS

BY

ROBERT CHARLES HANLON

B.S., Case Western Reserve University, 1968
M.S., University of Kansas, 1973

THESIS

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Chapter 1

INTRODUCTION

1.1. Spread Spectrum Communications Systems

A spread spectrum communications system is characterized by the use of much larger radio frequency bandwidths than required by the bandwidth of the baseband information signal [Dixon, 1976a], [Glazer, 1973]. By effective use of this additional bandwidth, spread spectrum systems develop several inherent advantages over systems which use conventional, relatively narrowband techniques. These advantages include [Cahn, 1973], [Dixon, 1976a], [Glazer, 1973]: (1) enhanced resistance to interference, (2) capability for multiple-access within a common frequency spectrum, (3) ability to operate effectively within a frequency spectrum simultaneously being occupied by conventionally modulated signals, (4) low detectibility by unintended receivers, and (5) message privacy. These benefits are interrelated and arise due to the methods employed to spread the bandwidth prior to transmission and compress (despread) the bandwidth after reception by the intended recipient. However, these techniques also give rise to two principal disadvantages: relatively complicated transmitting and receiving equipment, and conflict with current frequency allocation and assignment procedures.

There are several methods which are commonly used to spread the information signal bandwidth. These include direct sequence code modulation, frequency hopping, time hopping, swept frequency ("chirp") modulation, and hybrid

combinations of these methods. The basic underlying principle of spread spectrum systems is that the intended recipient knows both the technique employed and its specific implementation. The intended recipient is then able to use this knowledge to invert the spreading process, thereby despreading the received signal to reproduce the narrowband information signal. To clarify these concepts, direct sequence modulation is examined in detail. Further, the scope of the remainder of this paper is restricted to this form of spread spectrum modulation.

1.2. Direct Sequence Code Modulation

In a direct sequence (DS) code modulation system, the information signal (digital or analog) is multiplied by a spreading code sequence whose effective bandwidth is much greater than that of the information signal. This multiplication in the time domain is equivalent to convolution in the frequency domain. If the baseband information signal is nominally bandlimited to the interval $(-B_b, B_b)$ and the spreading code sequence is nominally bandlimited to the interval $(-B_m, B_m)$, then the product signal in general occupies the frequency interval $(-B_b-B_m, B_b+B_m)$. Since in spread spectrum systems, B_m is much greater than B_b, then the product signal bandwidth is approximately that of the spreading code sequence. For high rate spreading codes, the transmitted power spectral density can be made quite low. As a result of the low power spectral density, interference effects on conventionally modulated systems operating simultaneously over some small portion of the spread spectrum bandwidth can be minimal, and the detectibility of the spread spectrum

transmission by conventional receivers is significantly reduced. Low detectibility enhances message privacy and traffic flow security (precludes knowledge of message traffic volume) and compounds problems in radio direction finding.

For the purposes of this discussion, it is assumed that the information signal is a binary digital signal, b(t), and the spreading code sequence (sometimes referred to as a key sequence or signature sequence) is also a binary digital signal, a(t), where both signals can take the value of either plus or minus one during their respective bit intervals. The bit interval of the information signal is T_d and the bit interval for the spreading code sequence is T_c, where T_d is much greater than T_c to provide the required greater bandwidth for the spreading code. A bit of the spreading code is referred to as a chip to distinguish it from an information signal bit, consequently, T_c is referred to as the chip interval and T_d as the bit interval. The spreading code is repetitive with period LT_c, where L is the number of chips per period, and is referred to as the code length. It is assumed that the spreading code is synchronized to the information signal such that one period of the spreading code occurs during each bit, i.e.,

$$T_d = LT_c \quad (1.1)$$

The product signal is [a(t)b(t)]. If the product signal is multiplied by a(t), we have $[a^2(t)b(t)]$. Since a(t) is either plus or minus one, $a^2(t)$ is identically one, and $[a^2(t)b(t)]$ is equal to b(t), which is the original information signal. Therefore, the despreading operation is identical to the spreading operation, that is, multiplication of a signal by the spreading code. Clearly, the receiver requires prior knowledge of the spreading code. Hence, the spreading code must be deterministic. However, if the code length is fairly long, the spreading code appears to be random and is often referred to as a pseudorandom code sequence. The receiver must not only know the code sequence, but must also generate the sequence in-phase with the sequence which has been applied to the information signal. In a system employing radio propagation, a pure synchronous system is generally not possible. This is the reason for using a periodic sequence. The periodic nature of the code sequence enables the receiver to detect the phase and to synchronize to this phase.

If signals other than the intended spread spectrum signal are received, multiplication by the spreading code in the receiver spreads each of these undesired signals to at least the bandwidth of the spreading code. Narrowband filtering after the multiplication passes the despread intended signal while rejecting most of the power of unintended signals. This can be seen to be a correlation receiver where the correlation is against the spreading code of the intended signal. For conventional narrowband signals, their power spectral densities are reduced by a factor of approximately $(B_b/B_m)<<1$ by the receiver's despreading process. Even if the interference has the same bandwidth as the spread spectrum signal, its power spectral density would be reduced by approximately $(B_m/2B_m)=1/2$, that is, 3 dB. Further, a signal whose bandwidth

was on the order of the spread spectrum bandwidth would generally have an inherently low power spectral density. To help minimize noise and interference effects, a wideband filter can be used on the received signal prior to despreading to eliminate undesired components outside the spread spectrum bandwidth. It has also been shown that notch filters can be used to eliminate narrowband interference wihout undue distortion of the spread spectrum signal [Sussman and Ferrari, 1976].

1.3. Direct Sequence Multiple-Access Communications

As indicated, the despreading process is fairly effective against undesired signals. This feature can be used to obtain multiple-access capability by assigning each of the various system users a unique spreading code. Provided that the codes used do not exhibit a high degree of cross-correlation, then the intended receiver is able to effectively reject the signals from the other users. Use of codes to implement multiple-access communications is referred to as code division multiplexing and offers an attractive alternative to conventional time and frequency division multiplexing. The presence of the other users in code division multiplexing results in an inherent cochannel interference. To help minimize the cross-correlation between the assigned spreading codes and thereby reduce the cochannel interference, special codes such as Gold codes are generally used which are known to have good cross-correlation properties [Dixon, 1976a], [Gold, 1964], [Pursley, 1976]. It should be noted that when Gold codes or other linear codes are used, they do not provide message security in the sense of cryptographic codes [Dixon,

1976a], [Scholtz, 1977], [Ristenbatt and Daws, 1977], but do provide a degree of message privacy.

Figure 1 shows a basic direct sequence spread spectrum multiple-access (DS/SSMA) communications system [Gardner and Orr, 1979], [Orr, 1977]. The receiver is assumed to be frequency, phase and code synchronized to the intended signal (channel 1). Although acquisition and maintenance of synchronization is a major aspect of spread spectrum communications systems, it is beyond the scope of this paper and is assumed to be perfect in all subsequent analyses. Although the receiver is assumed to be synchronized to channel 1, the transmitters are operating asynchronously relative to each other.

1.4. Channel Model

In analyzing the performance of a communications system, the channel over which the signals propagate must be modeled to account for all significant sources of signal degradation. Along with the presence of additive random noise, it is relatively common in systems utilizing radio propagation to encounter random multiplicative signal distortion, that is, continuously time varying received signal amplitudes. This random multiplicative signal distortion is generally referred to as nonselective fading. Fading generally arises out of conditions which give rise to multipath propagation (e.g., ionospheric reflection or tropospheric scatter propagation) where the multiple paths are of varying lengths and nonstationary in nature (e.g., dynamic

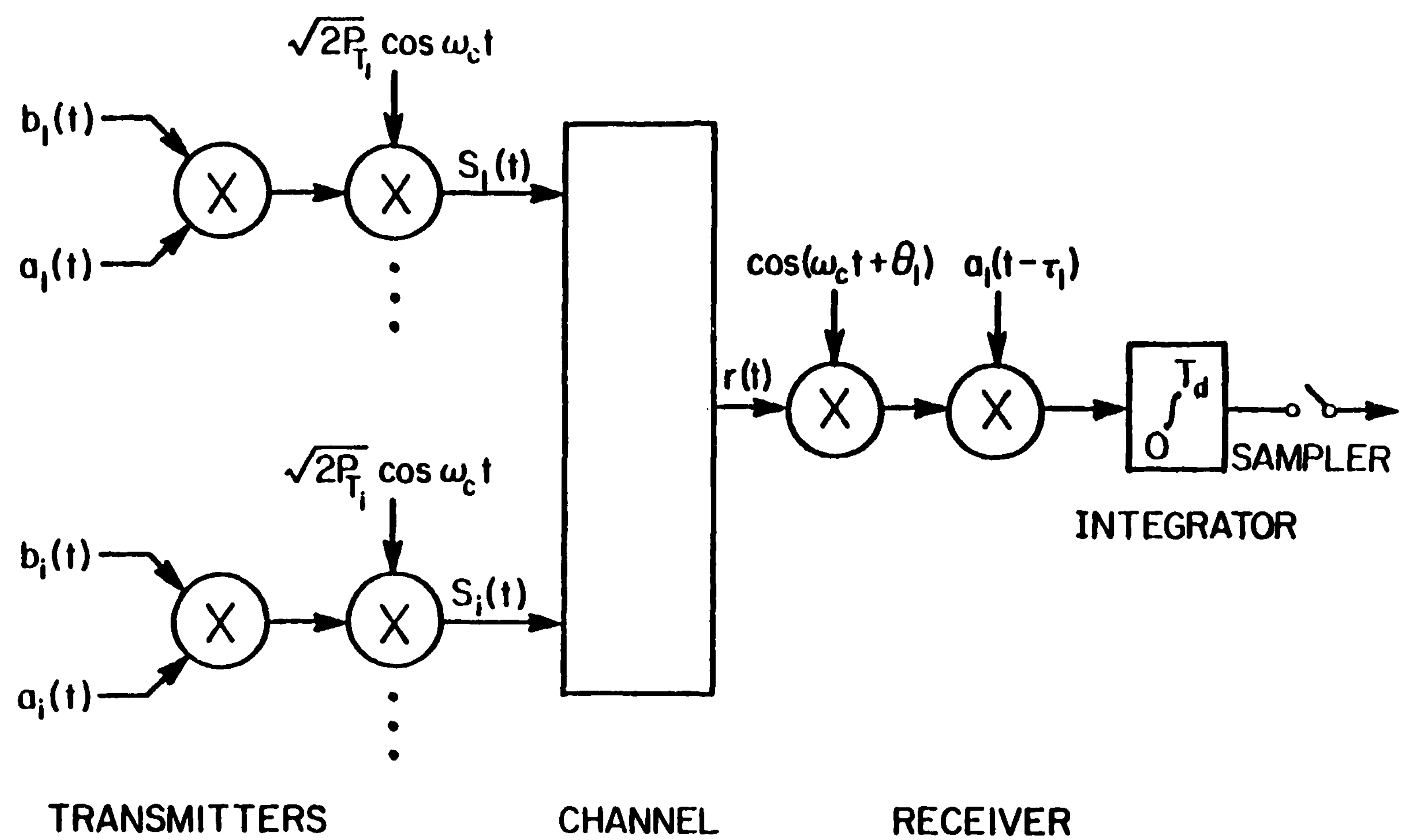

Figure 1. SSMA communications system diagram. b(t) is the data sequence and a(t) is the spreading code sequence

inhomogeneities in scattering volume). Nonselective fading implies that the fading effects do not change appreciably with frequency over the bandwidth of interest. In addition to assuming that the fading is nonselective, it is also assumed that the fading is slow , that is, the fading intensity does not change appreciably over time periods on the order of the data bit interval. For the purposes of this analysis, the additive random noise is assumed to be independent zero-mean white Gaussian noise (uniform two-sided power spectral density of $\eta_0/2$ with normally distributed amplitude). Figure 2 depicts the channel model [Gardner and Orr, 1979].

The fading intensities, $f_i(t)$, are assumed to be generalized Rayleigh variates (Appendix B). The generalized Rayleigh distribution encompasses a wide class of distributions which are useful in the study of communications over fading channels. In particular, the generalized Rayleigh distribution has as special cases: the Rician (specular plus Rayleigh), Nakagami (m-distributed), and Rayleigh distributions. Each of these distributions has been extensively utilized for modeling fading channels.

1.5. System Equations

For the spread spectrum multiple-access communications system depicted in Fig. 1, the transmitted signal on the i-th channel can be represented as an amplitude modulated carrier, that is, [Gardner and Orr, 1979]

$$S_i(t) = (2P_{Ti})^{1/2}\, a_i(t)\, b_i(t) \cos \omega_c t \qquad (1.2)$$

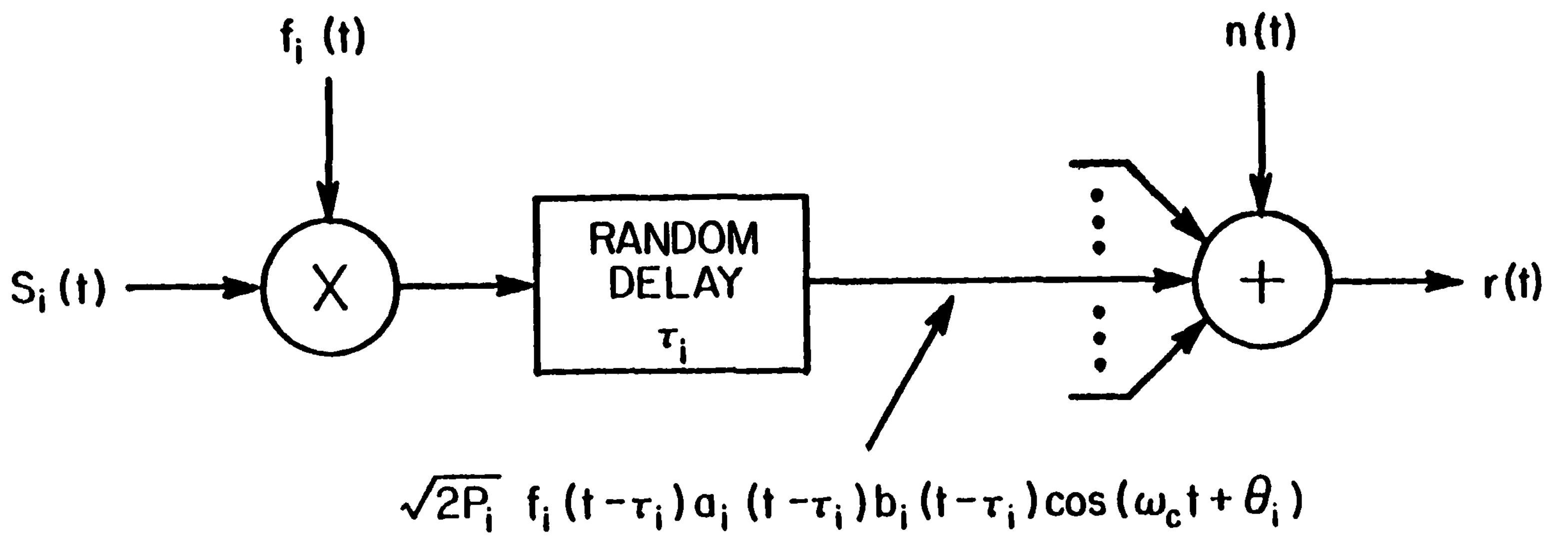

Figure 2. Channel model. $f_i(t)$ is the random fading envelope, τ_i is the random propagation delay, θ_i is the random carrier phase shift and $n(t)$ is the additive white Gaussian noise.

where P_{Ti} is the transmitted power, $a_i(t)$ is the spreading code, $b_i(t)$ is the information signal, and ω_c is the common carrier frequency. The channel model as illustrated in Fig. 2 introduces nonselective random fading and random path delays on each signal and random additive noise. The total received signal for the case of M users is then [Gardner and Orr, 1979]

$$r(t) = n(t) + \sum_{i=1}^{M} (2P_i)^{1/2} f_i(t-\tau_i)\, a_i(t-\tau_i)$$

$$b_i(t-\tau_i) \cos(\omega_c t + \Theta_i) \qquad (1.3)$$

where the P_i are the effective received powers in the absence of fading, the f_i are the random fading intensities, the τ_i are random time delays, and the Θ_i are random phase angles. The f_i, τ_i, and Θ_i are random processes; however, the time dependence is not explicitly shown to simplify the notation. The τ_i are assumed to be independent and identically distributed (i.i.d.) with uniform distributions on the interval $(0, T_d)$. The Θ_i are assumed to be i.i.d. with uniform distributions on the interval $(0, 2\pi)$. The assumption that the Θ_i are uniform implies the transmitters are operating asynchronously (as previously mentioned) and that the transmitters' relative phase drifts overshadow their phase variations due to fading effects.

For coherent reception and under the stated assumption that the intended receiver is perfectly synchronized (frequency, phase, and spreading code) to the first signal, detection is accomplished by multiplying the received signal by $[a_1(t-\tau_1) \cos(\omega_c t+\Theta_1)]$ and integrating over a data bit period, that is,

correlating the received signal with the spreading code of the intended signal. Since coherent detection is assumed, we can set τ_1 equal to zero, Θ_1 equal to zero, and restrict our interest to the data bit received during the interval $(0, T_d)$ without loss of generality. The carrier frequency is much greater than T_c^{-1}, hence the contribution from the double frequency components is negligible and the output of the correlation integrator is

$$R = b_1 S + I + N \tag{1.4}$$

where the intended signal component is

$$S = (P_1/2)^{1/2} \int_0^{T_d} dt\ f_1(t) \ , \tag{1.5}$$

the cochannel interference component is

$$I = \sum_{i=2}^{M} (P_i/2)^{1/2} \int_0^{T_d} dt\ a_1(t)a_i(t-\tau_i)b_i(t-\tau_i)f_i(t-\tau_i)\cos\Theta_i \ , \tag{1.6}$$

and the additive noise component is

$$N = \int_0^{T_d} dt\ a_1(t)\ n(t)\ \cos\omega_c t \ . \tag{1.7}$$

Under the assumption of slow fading, f_1, f_i, and Θ_i are relatively constant over the integration interval. We can then write

$$S = (P_1 T_d^2/2)^{1/2} f_1 \tag{1.8}$$

and

$$I = \sum_{i=2}^{M} (P_i T_d^2/2)^{1/2} c_{n_{1i}} f_i \cos\theta_i \tag{1.9}$$

where $c_{n_{1i}}$ is the normalized cross-correlation between codes a_1 and a_i in the presence of data, that is,

$$c_{n_{1i}} = T_d^{-1} \int_0^{T_d} dt\ a_1(t) a_i(t-\tau_i) b_i(t-\tau_i) \ . \tag{1.10}$$

For equally likely data bits, the detection threshold is zero due to symmetry, and the probability of error is

$$\Pr\{E\} = \frac{1}{2} [\Pr\{S+I+N<0 | b_1=1\} + \Pr\{-S+I+N>0 | b_1=-1\}] \ . \tag{1.11}$$

By letting $(Z = I+N)$ and by noting that the probability density of Z is symmetric, we obtain for the average probability of error

$$\Pr\{E\} = \int_0^\infty d\omega \int_0^\omega d\nu\ p_{sz}(\nu,\omega) = \frac{1}{2} \Pr\{|Z| > S\} \tag{1.12}$$

where p_{sz} is the joint probability density of the random processes S and Z.

The expression for the probability of error given by Eq.(1.12) appears quite simple; however, determination of the joint probability density p_{sz} and subsequent evaluation of the integrals is in general quite complicated due to the numerous random processes involved. Three general approaches have been used to evaluate or approximate the system performance. The first involves determination of the equivalent signal-to-noise ratio (SNR) resulting from the cochannel interference and additive noise, that is, [Gardner and Orr, 1979], [Pursley, 1977]

$$SNR = \langle S^2 \rangle / Var(Z) . \tag{1.13}$$

The equivalent SNR can then be taken as an indirect measure of the system performance (bit error rate) or an estimate of the probability of error can be developed by using the equivalent SNR in the well-known error expression for signals in Gaussian noise

$$Pr\{E\} = \frac{1}{2} erfc[(SNR/2)^{1/2}] \tag{1.14}$$

where erfc(x) is the complementary error function. The second approach involves bounding the error probability by utilizing the theory of moment spaces [Borth, 1979], [Yao, 1976 and 1977]. The third approach, and that addressed in the remainder of this paper, is to develop a series expansion for the error rate in the cross-correlation moments of the spreading codes [Gardner and Orr, 1979], [Hanlon and Gardner, 1979 and 1980]. Provided that the series expansion converges rapidly, the first few terms provide an accurate estimate

of the average error rate.

1.6. Outline of the Thesis

As indicated in the preceding section, the approach used in this paper to evaluate the error performance is the use of series expansions of the error rate in the cross-correlation moments of the spreading codes. To utilize this approach, we require access to the code cross-correlation moments. This problem is addressed first.

As seen later, in general, evaluation of the cross-correlation moments requires knowledge of the cross-correlation probability density. Obtaining the probability density can be very difficult, and to circumvent this problem, accurate estimates of the moments are desired. In Chapter 2, an algorithm is developed which can be used to evaluate the probability density from the joint probability mass function for adjacent cross-correlation values for integral offset. This algorithm is used to evaluate the moments for the cross-correlation for a set of Gold codes of length 31 and a subset of Gold codes of length 127. Gold codes are used throughout the analysis as representative spreading codes for illustration. These moments and the moments of random codes of the same length (Appendix D) are used to develop an empirical estimate for the low-order cross-correlation moments for Gold codes based on modeling the cross-correlation distribution by a generalized gamma distribution. The empirical estimates developed from the gamma density model are used in the estimates of the probability of error developed in Chapter 3.

The error performance analysis in Chapter 3 addresses two special cases. First, the case of a single cochannel interferer in the absence of additive noise. This case represents a performance bound for multiple-access communications, that is, it represents an irreducible error rate in the presence of cochannel interference and fading for arbitrarily large signal-to-Gaussian-noise ratios. The second case addressed is for multiple independent interferers with additive Gaussian noise. This case also gives rise to an irreducible error rate for arbitrarily large signal-to-Gaussian-noise ratios. As part of the analysis, simple single-term estimates for these irreducible error rates are developed and evaluated.

In both cases mentioned above, the analytical results are based on the developments given in Appendix B for generalized Rayleigh variates. In particular, the probability of error is expressible in terms of the expressions for the probability that one generalized Rayleigh variate exceeds another for either independent or correlated variates.

Chapter 4 summarizes, provides conclusions developed from the results presented earlier, and gives recommendations for further study in this area.

Throughout the paper, reference is made to mathematical functions and formulas which are listed in Appendix A.

Chapter 2

CROSS-CORRELATION BETWEEN SPREADING CODES

2.1. Cross-Correlation in the Presence of Data

In a direct sequence spread spectrum communications system, multiple-access capability within a common frequency spectrum is obtained by assigning each system user a unique periodic spreading code. The presence of multiple users within a shared frequency spectrum results in an inherent cochannel interference. However, if the cross-correlation between the spreading codes is kept low, then the cochannel interference effects are relatively minimized (subject to constraints such as number and proximity of other users) and the likelihood of reliable communications is enhanced.

For direct sequence multiple-access systems, the output of the correlation receiver, which is assumed to be synchronized to the intended signal, contains cochannel interference terms of the form [Gardner and Orr, 1979], [Pursley, 1977]

$$L\, T_d^{-1} \int_0^{T_d} dt\; a_j(t-\phi_j)\; a_i(t-\tau_i-\phi_i)\; b_i(t-\tau_i) \tag{2.1}$$

where a_j is the spreading code on the intended signal, a_i and b_i are the spreading code and data bit respectively on the i-th interfering user's signal, ϕ_j and ϕ_i set the phase of the spreading codes relative to their data bits, L

is the length of the spreading codes (number of chips in one period of the code), T_d is the data bit interval (assumed to be identical for all users), and τ_i is the relative timing offset in the receiver between the intended user's and interfering user's signals. This timing offset is due to asynchronism between transmitters as well as differing propagation delays over the channel. The τ_i are assumed to be uniformly distributed on the interval $(0,T_d)$. Both the spreading codes and data signal can assume values of +1 or -1. The spreading codes have a period of L chips where each chip is of duration T_c. This period is assumed to correspond to one data bit interval.

The integral in Eq.(2.1) is similar to the cross-correlation between the two spreading codes except for the presence of the interfering signal's data bit. Consequently, it is referred to as the cross-correlation in the presence of data and designated $\hat{c}$. The factor b_i considerably complicates analysis of such forms. Although special sets of spreading codes which have good cross-correlation properties (typically meaning bounded by some value much less than L) are generally used, the presence of the data bit significantly alters these properties [Massey and Uhran, 1975]. For instance, the magnitude of cross-correlation values for Gold codes [Gold, 1964] of length 31 never exceeds 9. However, with data present, the magnitude of the integral in Eq.(2.1) for the same codes can be as large as 21 [Orr, 1977], [Massey and Uhran, 1975].

In evaluating the multiple-access system performance, two limiting cases are of particular interest: the case of a single interferer and the case where there are several simultaneous interferers. When referring to a single interferer, this does not imply that there is only one other system user, but

rather that only one of the other system users is simultaneously transmitting (this case would typically occur if the various users each had a low transmission duty cycle). Likewise, in the case of several interferers, say M-1, there are in general more system users than M. However, there are only M system users (M-1 interferers) simultaneously transmitting. In the case of several interferers, the composite interference is a sum of integrals of the form given by Eq.(2.1). Since both the data sequences and timing offsets are independent, the interference in the multiple interferer case tends toward a Gaussian random process. In either case, the probability of error is a function of $\hat{c}$. To exactly evaluate the error rate, the probability density of $\hat{c}$ would be needed. However, as shown later, determination of the probability density can be quite complicated. For a single interferer, the problem of determining the probability density can sometimes be circumvented by expanding the conditional probability of error given $\hat{c}$ in a series involving the powers of $(\hat{c}/L)$. An estimate of the error rate can then be obtained by truncating the series and taking the expectation. This approach has been shown to be quite useful for estimating error rates of direct sequence systems operating over nonselective slow fading channels [Gardner and Orr, 1979], [Hanlon and Gardner, 1979 and 1980]. For the case of multiple interferers, if we approximate the interference by a Gaussian random process, we can often restrict our interest to determination of the second moment of the cross-correlation. Therefore, in many cases the probability of error can be accurately estimated by determining the first few moments of $\hat{c}$ or, more practically, an estimate of these moments.

For a given multiple-access communications system, the code that is designated a_j depends on which user is taken to be the intended user for the analysis. Likewise, a_i depends on the signal whose contribution to the cochannel interference is to be determined. In general, it could be any of the other users. From a systems viewpoint, we are concerned with the performance of the system over all possible combinations of users. Further, the codes assigned to any or all users might be occasionally changed due to operational considerations (e.g., use of dynamic code assignments to enhance the long-term privacy or antijam characteristics). Consequently, exact analysis for a particular pair of codes is of limited utility, and we are concerned primarily with the expected performance (average error rate) over the entire set of permitted codes. The analysis then addresses the average over all possible users (i.e., a_j and a_i are assumed to be equally likely to be any of the possible code sequences, where i is not equal to j). This, coupled with the presence of the randomizing effects due to the interfering signal's data sequence and the relative timing offsets (τ_i), leads to the conjecture that the low-order moments of the average cross-correlation in the presence of data is close to those for random code sequences of the same length (L chips per data bit). This conjecture is heuristically tested by example.

In Section 2.2 an algorithm for calculating the probability density for the code cross-correlation in the presence of data is developed. Although our immediate interest is in determining the average over all possible users (entire code set), the algorithm can be directly applied to any subset of the codes if, in a specific application, a priori considerations warrant such a restriction. This probability density can then be used to obtain exact moments

for whatever code set or subset is under consideration. The probability density is used to determine the moments of the code cross-correlation in the presence of data for a set of Gold codes of length 31 and for a subset of Gold codes of length 127.

In Section 2.3 two approaches for estimating the low-order Gold code moments are addressed. The first method directly estimates the low-order moments by the corresponding moments for random code sequences of length L (Appendix D). The second method empirically refines these estimates by using the moments of a generalized gamma density for which the parameters are chosen to provide close agreement with the Gold code moments.

2.2. Probability Density

In Eq.(2.1), the product $a_j a_i b_i$ is not periodic due to the presence of the factor b_i. However, since the correlator is restricted to the interval $(0,T_d)$, the integrand can be considered to be a sample of the product of periodic codes defined respectively by a_j and the product $a_i b_i$ on the integration interval. The cross-correlation between a pair of periodic synchronous codes is given by the number of agreements minus the number of disagreements when the codes are compared chip-by-chip over one period of the codes [Gold, 1964], [Dixon, 1976a]. For the codes to be synchronous, we require the codes to have the same period and the same chip intervals and for the chip intervals to coincide in time. For a pair of codes of length L (period of L chips), we have

$$c = 2A - L \tag{2.2}$$

where c is the cross-correlation value and A is the number of agreements. Hence, c is even for L even and odd for L odd (c mod2 = L mod2) and is restricted to the closed interval [-L, L].

For a_j and the product $a_i b_i$ to be synchronous as defined above, we must restrict τ_i to be of the form

$$\tau_i = n\, T_c \tag{2.3}$$

where n is equally likely to take any integer value on the closed interval [0, L-1]. We designate the random variable for the cross-correlation for integral offset by c. The probability mass function for c is nonzero only for c = m where m satisfies $-L \leq m \leq L$ and m mod2 = L mod2. For an arbitrary timing offset, τ_i can be represented as

$$\tau_i = (n + x)T_c \tag{2.4}$$

where again n is equally likely to take any integer value on the closed interval [0, L-1] and x is uniformly distributed on the interval (0,1). In this case, the code cross-correlation can take noninteger values due to partial agreements and disagreements during the chip intervals. The case of arbitrary timing offsets (to include nonintegral offsets) is more realistic since, in general, it is neither desirable nor feasible to impose any synchronization on

the system, due to the complexity and cost which would be involved. Let c_0 be a random variable corresponding to the value of the cross-correlation when $x = 0$, and c_1 be the value when $x = 1$. For given values of c_0 and c_1, the cross-correlation varies linearly with x as each chip comparison transitions from its value for c_0 to its value for c_1. Consequently, the cross-correlation for arbitrary offset, designated $\hat{c}$, is given by

$$\hat{c} = (1 - x)c_0 + xc_1 \ . \tag{2.5}$$

When $c_0 = c_1$, $\hat{c}$ takes the value c_0 for all values of x. Hence, the probability of $\hat{c}$ taking the value k is equal to the probability that both c_0 and c_1 take the value k, that is,

$$\Pr\{\hat{c} = k\} = \Pr\{c_0 = k,\ c_1 = k\} = P'_{k,k} \tag{2.6}$$

where $P'_{r,s}$ is the joint probability mass function for the ordered pair ($c_0 = r$, $c_1 = s$). To obtain the density for $\hat{c}$, we first obtain the conditional density of $\hat{c}$ given c_0 and c_1. Since x is uniform on (0,1), we have

$$p_{\hat{c}}(\lambda | c_0, c_1) = \begin{cases} |c_1 - c_0|^{-1} \ , & \min(c_0,c_1) < \lambda < \max(c_0,c_1) \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{2.7}$$

The density is then obtained by taking the expectation of the conditional

density. Since the conditional density is constant, the density of $\hat{c}$ is piecewise constant with discontinuities occurring whenever $\lambda = k$ for $-L \le k \le L$ and $k \bmod 2 = L \bmod 2$. For λ on the open interval $(k,k+2)$ where k is even for L even and odd for L odd, we have $(r \bmod 2 = s \bmod 2 = L \bmod 2)$

$$p_{\hat{c}}(\lambda) = \sum_{r=-L}^{k} \sum_{s=k+2}^{L} (s-r)^{-1} P'_{r,s} + \sum_{r=k+2}^{L} \sum_{s=-L}^{k} (r-s)^{-1} P'_{r,s} \tag{2.8}$$

where any open sums are to be interpreted as zero. This equation can then be manipulated into the form [Hanlon, Peterson, and Gardner, 1980]

$$p_{\hat{c}}(\lambda) = \sum_{r=0}^{(L+k)/2} \sum_{s=(L+k+2)/2}^{L} (s-r)^{-1} P_{2r-L,2s-L} \tag{2.9}$$

where $(k < \lambda < k+2,\ -L \le k \le L-2,\ k \bmod 2 = L \bmod 2)$ and

$$P_{r,s} = \frac{1}{2} (P'_{r,s} + P'_{s,r}) \ . \tag{2.10}$$

It should be noted that for the joint probability mass function $P'_{r,s}$, the order of r and s is important since r designates the value of the cross-correlation for $x = 0$ (c_0) and s designates the value for $x = 1$ (c_1), whereas, in Eq.(2.9), the result is independent of the order since $P_{r,s}$ is the average of $P'_{r,s}$ and $P'_{s,r}$. Consequently, $P_{r,s}$ can be considered to be the joint probability mass function of c_0 and c_1 without regard to order.

The relations shown in Eq.(2.6) and Eq.(2.9) demonstrate that the distribution of $\hat{c}$ is dependent only on the joint probability mass function for adjacent cross-correlation values under the condition of integral offset. This significantly reduces the complexity of calculating the density for arbitrary offset between the codes. For example, consider the case of random binary symmetric codes, that is, codes for which the value of each chip is equally likely to be either +1 or -1 and for which each chip is statistically independent of all other chips. In this case, for r mod2 = s mod2 = L mod2,

$$P_{r,s} = 2^{-2L} \binom{L}{\frac{L+r}{2}} \binom{L}{\frac{L+s}{2}} \tag{2.11}$$

and from Eq.(2.6) and Eq.(2.9), we have for the mass function

$$\Pr\{\hat{c} = m\} = 2^{-2L} \binom{L}{\frac{L+m}{2}}^2 \tag{2.12}$$

where ($-L \leq m \leq L$, m mod2 = L mod2), and for the density

$$p_{\hat{c}}(\lambda) = 2^{-2L} \sum_{r=0}^{(L+k)/2} \sum_{s=(L+k+2)/2}^{L} \binom{L}{r} \binom{L}{s} (s-r)^{-1} \tag{2.13}$$

where ($k < \lambda < k+2$, $-L \leq k \leq L-2$, k mod2 = L mod2). Note that in the case of random codes, the magnitude of the cross-correlation can be as large as L, the code length (number of chips per data bit).

For a general set of periodic codes, determination of the probability density is more complicated. Starting with Eq.(2.1) and noting that the data bit on the interfering signal has a constant value, say b_i, on the interval $(0,\tau_i)$ and transitions to a constant value, say $\hat{b}_i$, for the remainder of the integration interval, we have

$$c = b_i \, R_{ji}(\tau_i) + \hat{b}_i \, \hat{R}_{ji}(\tau_i) \tag{2.14}$$

where R_{ji} and $\hat{R}_{ji}$ are partial cross-correlation functions defined by [Pursley, 1977]

$$R_{ji}(\tau_i) = T_c^{-1} \int_0^{\tau_i} dt \; a_j(t-\phi_j) \, a_i(t-\phi_i-\tau_i) \tag{2.15}$$

$$\hat{R}_{ji}(\tau_i) = T_c^{-1} \int_{\tau_i}^{T_d} dt \; a_j(t-\phi_j) \, a_i(t-\phi_i-\tau_i) \; . \tag{2.16}$$

Since $(b_i, \hat{b}_i)$ can assume any of the four values $(\pm 1, \pm 1)$ with equal likelihood, then for a given offset and phase, c takes on the four values

$$c = \pm[R_{ji}(\tau_i) \pm \hat{R}_{ji}(\tau_i)] \tag{2.17}$$

with equal likelihood. It is readily shown by a change of variables (see also Massey and Uhran [1975]) that these four values are the same as

$$c = \pm[R_{ij}(T_d - \tau_i) \pm \hat{R}_{ij}(T_d - \tau_i)] \ . \tag{2.18}$$

Then, for a given set of phases for a_j and a_i, varying the timing offset over all possible integral values generates the same values for the cross-correlation, probability mass function, and probability density whether a_j is taken to be the intended signal's code sequence and a_i is taken to be the interfering signal's code sequence or vice versa.

Note, since the integration interval implied in Eq.(2.14) is broken, that the set of values for the partial cross-correlation functions under varying integral offsets changes if either code sequence is circularly shifted by q chips where q lies on the interval [0,L-1]. This circular shift is equivalent to the code sequence being started with a *different phase* relative to its data bit sequence. In principle, we could select the phase for each code to minimize the average cross-correlation moments or average bit error rate. However, for long codes this is impractical, and we do not assume any particular set of phases for the codes, that is, the relative phase is assumed to take on any of the L possible values with equal likelihood. We can determine the values of $P'_{r,s}$ by tabulating the relative frequency of the ordered pair (c_0,c_1), where c_0 corresponds to $\tau_i = nT_c$ and c_1 corresponds to $\tau_i = (n+1)T_c$, under exhaustive enumeration of the partial cross-correlation functions over all values of n on the interval [0,L-1]; over all code pairs (a_j,a_i), for i greater than j; and over all phases of the codes. The symmetry of Eq.(2.17) and Eq.(2.18) obviates consideration of the cases where i is less than j. $P_{r,s}$ is then the average of $P'_{r,s}$ and $P'_{s,r}$. The values of $P_{r,s}$ are then used in Eq.(2.6) and Eq.(2.9) to obtain the probability density.

For the case of Gold codes of length L, there are (L+2) distinct code sequences which can be generated by a linear shift register with 2m stages, where $L = (2^m - 1)$. For example, there are 33 possible codes of length 31 which can be generated by a 10 stage shift register. Let $a_j^{(0)}$ represent the nominal zero phase of code a_j relative to its data bit and $a_j^{(k)}$ represent a phase shift of k chips relative to $a_j^{(0)}$. From the closure property of linear shift registers (the modulo 2 sum of two sequences from a linear shift register is itself a sequence from that shift register)[Gold, 1964] and the isomorphism between the additive group {0,1} and the multiplicative group {-1,1}, it is readily shown that the chip-by-chip product $a_j^{(k)}a_i^{(m)}$, for $a_j^{(k)}$ not equal to $a_i^{(m)}$, can always be resolved as the product of $a_p^{(0)}$ and $a_r^{(s)}$, that is, the product of the nominal zero phase of one of the Gold codes and some other member of the set of Gold codes. The values for $P_{r,s}$ generated by the code combination $a_j^{(k)}a_i^{(m)}$ then replicate the set of values generated by the code combination $a_p^{(0)}a_r^{(s)}$. Consequently, we only consider phase shifts of the interfering signal's code sequence rather than phase shifting both the desired and interfering signals' code sequences.

Calculation of $P_{r,s}$ for Gold codes then involves $\binom{L+2}{2}$ code pairs, L different values for the relative timing offset, L different values for the relative phase of a_i, and 4 different data bit combinations. Hence there are $[2L^2(L+1)(L+2)]$ values for the cross-correlation. Due to symmetry, only half of these values need be calculated. Consequently, the number of cross-correlation values calculated is proportional to the fourth power of the code length. Figure 3 illustrates the results for a set of Gold codes of length 31 (m=5). Without the presence of the data bits, the cross-correlation between Gold codes

of length L is three-valued: $(-2^{\lfloor (m+2)/2 \rfloor} - 1,\ -1,\ 2^{\lfloor (m+2)/2 \rfloor} - 1)$ [Gold, 1964]. Figure 3a shows these three values (-9, -1, 7) and the average relative frequency with which they occur for integral offsets for the Gold codes of length 31. The results shown in Fig. 3b illustrate the effects of data on the cross-correlation values for integral offset. Note that the probability mass function is symmetric and the maximum magnitude of the cross-correlation (21) extends considerably beyond the corresponding value without data present (9). Figure 3c shows the average cross-correlation probability density and probability mass function for arbitrary offset. As mentioned in the development for arbitrary code sets, the density is piecewise constant.

2.3. Moments

Given the probability density, it is a simple matter to calculate the moments of the cross-correlation. However, if we desire the average performance, the number of code correlations needed to compute the density rapidly becomes prohibitive as the code length increases. Hence, we are motivated to develop an approximation to the average error rate which can be evaluated without use of the probability density. One approach is to expand the conditional probability of error given $\hat{c}$ in a power series of the form

$$\Pr\{E|\hat{c}\} = \sum_{k} A_k (\hat{c}/L)^{2k} \tag{2.19}$$

where the A_k are the coefficients of the expansion. Provided such a series exists and that it converges sufficiently rapidly, the expectation of the first

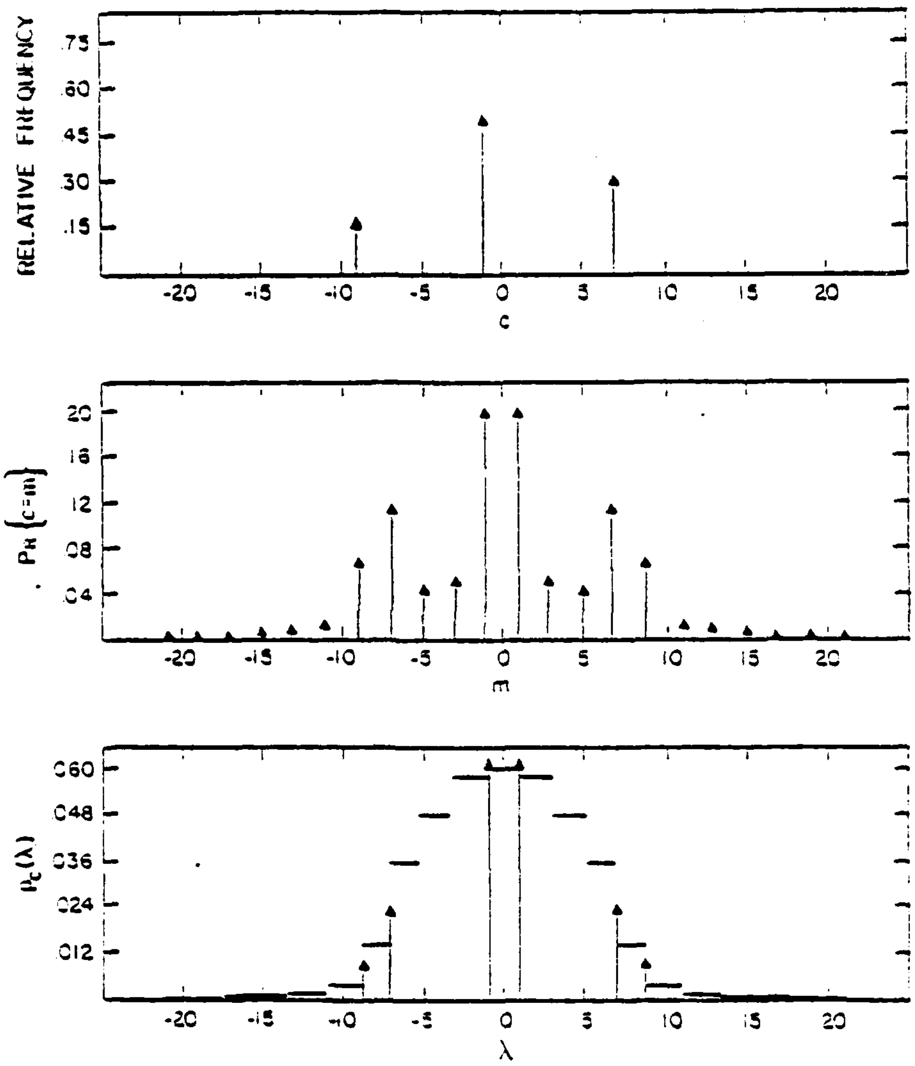

Figure 3. Probability mass functions and density for cross-correlation of Gold codes of length 31: (a) relative frequency of values for integral offset, without data; (b) mass function for integral offset, with data; (c) mass function and density for arbitrary offset, with data.

few terms provides an accurate estimate of the error rate. This would limit the problem to determining the first few moments of the cross-correlation or, more practically, an estimate of these moments. This approach has been shown to provide accurate estimates of the probability of error for direct sequence systems operating over nonselective slow fading channels [Gardner and Orr, 1979], [Hanlon and Gardner, 1979 and 1980].

It has been previously conjectured that the low-order moments of the average cross-correlation in the presence of data are close to those for random code sequences of the same length. It is shown in Appendix D that the cross-correlation moments of order 2n for random code sequences can be expressed as a polynomial of degree n in the code length. Table D.2 lists the coefficients of these polynomials for the first five even moments. For large values of L (long codes), the moments are shown to be approximately given by

$$\langle \hat{c}^{2n} \rangle \simeq \frac{(2n-1)!!}{2n+1} L^n \sum_{m=0}^{n} \binom{n}{m} \binom{2n}{2m}^{-1} \{1 - [m^2+(n-m)^2-n]/3L\} . \tag{2.20}$$

This expression is exact for the second and the fourth moments which are, respectively, $(2L/3)$ and $(7L^2/5 - 4L/5)$. These values differ significantly from the corresponding moments for integral offset, which are L and $(3L^2 - 2L)$ (Table D.1). Consequently, when analyzing the performance of nonsynchronous systems, estimates of the probability of error or related functions (e.g., signal-to-noise ratio) which make use of the cross-correlation moments, must consider arbitrary offsets rather than just integral offsets.

Tables 1 and 2 show the low-order moments of random codes and Gold codes for code lengths of 31 and 127, respectively. As anticipated, the corresponding moments are fairly close, particularly the second moments. However, as the order of the moments increases, the values begin to differ significantly. The randomizing effects of the interfering signal's data sequence and the relative timing offset cause the density of the cross-correlation for Gold codes in the presence of data to take a similar form to that of random code sequences of the same length. One significant difference is the maximum value of the cross-correlation magnitude. The effect of this difference on the second moment for Gold codes relative to random codes is small due to the low probability that the cross-correlation for random codes exceeds the maximum value for Gold codes. However, as the order of the moments increases, the outlying values are weighted more in the calculation of the moments and the difference in values for the moments is progressively more pronounced.

To facilitate development and use of the empirical estimates, it is desirable to consider a continuous density. If we develop a close approximation to the low-order Gold code moments by use of the moments for a continuous density, we would, in principle, be able to extrapolate the approximation to provide estimates of the low-order absolute moments or low-order nonintegral moments of the Gold codes. These generalized moments are of potential use when the series expansion of the conditional probability of error given $\hat{c}$ has a more complicated dependence on $\hat{c}$ than that shown in Eq.(2.19).

TABLE 1. CROSS-CORRELATION MOMENTS $\langle \hat{c}^{2n} \rangle$ (L = 31)

n	Gold Codes	Random Codes	Gamma Density
1	2.05 E 1	2.07 E 1	2.07 E 1
2	1.12 E 3	1.32 E 3	1.15 E 3
3	9.24 E 4	1.45 E 5	9.95 E 4
4	1.08 E 7	2.27 E 7	1.15 E 7
5	1.73 E 9	4.64 E 9	1.65 E 9

TABLE 2. CROSS-CORRELATION MOMENTS $\langle \hat{c}^{2n} \rangle$ (L = 127)

n	Gold Codes	Random Codes	Gamma Density
1	8.75 E 1	8.47 E 1	8.47 E 1
2	1.97 E 4	2.25 E 4	1.93 E 4
3	6.58 E 6	1.04 E 7	6.84 E 6
4	2.96 E 9	6.99 E 9	3.24 E 9
5	2.04 E12	6.26 E12	1.90 E12

The symmetry and general shape of the density for $\hat{c}$ (Fig. 3) suggest a Gaussian approximation. The main disadvantage of using a Gaussian density is that the range of the density is unbounded (tails extend indefinitely). This tends to exacerbate the effect of the difference in the magnitude of the maximum cross-correlation values. However, the density falls off fairly rapidly and would provide reasonable approximations to the low-order moments, which are of prime concern to our application. The effect of the tails can be reduced if the Gaussian density is generalized slightly to increase the rate at which the tails decay. One generalization of the Gaussian density is the two-sided generalized gamma density. The form of the generalized gamma density is given by

$$p(x) = A|x|^{\alpha-1} \exp(-\beta|x|^{\nu}) \tag{2.21}$$

where for the Gaussian density, α equals one and ν equals two. By keeping α equal to one and setting the second moment equal to that of the random codes, i.e., 2L/3, the moments for the gamma density are then given by

$$\langle x^{2n} \rangle = [\Gamma(\frac{2n+1}{\nu})/\Gamma(\frac{1}{\nu})]\ [2\Gamma(\frac{1}{\nu})/3\Gamma(\frac{3}{\nu})]^n\ L^n\ . \tag{2.22}$$

To develop the empirical estimates, the value of ν is then chosen to approximately match the first five even moments of the gamma density to the corresponding moments of the cross-correlation for Gold codes of length 31. The appropriate value of ν is about 2.4. It is expected that this choice of ν would also result in a good estimate of the low-order moments for Gold codes

of longer lengths due to (1) anticipated close agreement of the second moments, regardless of code length, and (2) the dependence on code length for these estimates is the same as in the single-term approximation for the cross-correlation moments for random codes. Table 1 shows the moments for the average cross-correlation for Gold codes of length 31, the corresponding moments for random sequences of the same length and moments of the generalized gamma density ($\nu = 2.4$). The moments of the generalized gamma density are seen to agree very well with the moments of the Gold codes.

For Gold codes of length 127, the moments were calculated using a subset of 5 out of the possible 129 Gold codes (permitting 10 pairwise combinations out of a possible 8256 combinations). We are still concerned primarily with the average over the entire code set; however, to reduce the calculations, we have used this subset to approximate the desired results. Table 2 shows the corresponding moments for this subset. The same general pattern appears as in the case of Gold codes of length 31. The agreement in the second moments is very good considering the small sample used.

The random code moments are of particular interest in related applications. In order to take full advantage of the potential benefits of spread spectrum systems, very long code sequences are desirable. These long code sequences enhance the privacy and antijam characteristics. For a given type of code, the longer the code sequence, the harder it is to effectively monitor or jam the communications system on which it is used. However, there are practical limits (technical and economic) to arbitrarily expanding the transmitted bandwidth [Dixon, 1976a]. In some applications then, the period of the spreading codes

is chosen to be much greater than the data bit interval. This provides for both the maximum feasible degree of spectrum spreading and very long code sequences. Let L be the period of the spreading codes and L' be the number of chips per data bit, where L is much greater than L'. Then if the spreading codes exhibit a high degree of randomness [Golomb, 1967] over a data bit interval, such as in the case of cryptographic codes, we would expect that the cross-correlation moments would be very closely approximated by the moments of random codes of length L'.

Chapter 3

PROBABILITY OF ERROR

3.1. Single Interferer Without Additive Noise

For the case of a single interferer where the additive noise is negligible, the probability of error represents an irreducible error rate which bounds the multiple-access system performance. Again, the restriction of consideration to a single interferer does not imply that there is only one other system user, but rather that only one of the other system users is simultaneously transmitting. In this case, from Eq.(1.12), the probability of error is given by

$$\Pr\{E\} = \frac{1}{2} \Pr\{af_2 > f_1\} \tag{3.1}$$

where

$$a^2 = (P_2/P_1)\,|\hat{c}_n \cos\Theta|^2 \quad , \quad \hat{c}_n = (\hat{c}/L) \quad , \quad \Theta = \Theta_2 \,. \tag{3.2}$$

It should be noted that f_1, f_2, $\hat{c}_n$, and Θ are all random. To facilitate the evaluation of this expression, the conditional probability of error given the factor a is first determined.

It is readily shown that if f has a generalized Rayleigh distribution, that is, $f \sim R(m,\beta,\Omega)$, then the product af also has a generalized Rayleigh distribution: $af \sim R(m,a\beta,a^2\Omega)$. From Appendix B, Eq.(B.26), we have

$$\Pr\{E|a\} = \frac{1}{2}(1-\rho^2)^m \exp(-m\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} (a^2\Omega_2/\Omega_1)^{k+m}$$

$$(-1)^k \; G_k(m,D_1^2/B_1,D_2^2/B_2,\rho^2) \tag{3.3}$$

where G_k is given by Eq.(B.27) and B_1, B_2, D_1, and D_2 are given by Eq.(B.21). It should be noted that the function G_k has no dependence on the factor a. The probability of error is then obtained by taking the expectation of the conditional probability of error with respect to both $\hat{c}_n$ and Θ which are independent of each other. From the definition of the beta function and the assumption that Θ is uniform on the interval $(0,2\pi)$, we readily obtain

$$\langle|\cos\,\Theta|^{2k+2m}\rangle = B(1/2,k+m+1/2)\,/\,\pi = \binom{m-1/2}{m}\frac{(m+1/2)_k}{(m+1)_k} \;. \tag{3.4}$$

By applying this result to Eq.(3.3), we have [Hanlon and Gardner, 1980]

$$\Pr\{E\} = \frac{1}{2}\binom{m-1/2}{m}(1-\rho^2)^m \exp(-m\beta_1^2/\Omega_1) \sum_{k=0}^{\infty}\frac{(m+1/2)_k}{(m+1)_k}(-1)^k$$

$$(P_2\Omega_2/P_1\Omega_1)^{k+m} \langle|\hat{c}_n|^{2k+2m}\rangle \; G_k(m,D_1^2/B_1,D_2^2/B_2,\rho^2) \;. \tag{3.5}$$

The bivariate density given by Eq.(B.18) implicitly assumes that the m factors are the same for both fading intensities. If we assume that the fading intensities are independent, so that the joint density is given by the product of the marginal densities, we can readily extend the results to the case where the m factors are different (Eq.(B.11)) [Hanlon and Gardner, 1980]

$$\Pr\{E\} = 2^{m_1-m_2-1} \binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} \frac{(m_1+1/2)_k}{(m_1+1)_k} (-1)^k$$

$$(m_1P_2\Omega_2/m_2P_1\Omega_1)^{k+m_1} \langle|\hat{c}_n|^{2k+2m_1}\rangle\, L_k^{(m_1-1)}(m_1\beta_1^2/\Omega_1)\, L_{k+m_1}^{(m_2-1)}(-m_2\beta_2^2/\Omega_2) \;. \quad (3.6)$$

The practical usefulness of Eq.(3.5) or Eq.(3.6) depends on the series converging sufficiently rapidly to enable close approximation of the average probability of error by one or two terms of the series. Since the cross-correlation moments for the spreading codes are typically very small for well-designed DS/SSMA systems, rapid convergence is quite feasible provided the ratio $(m_1P_2\Omega_2/m_2P_1\Omega_1)$ is not large. Table 3 lists the single-term approximations for the more important special cases of the generalized Rayleigh distribution.

Numerical evaluation of the average probability of error requires determination of the absolute moments of the normalized code cross-correlation. This can be very difficult, particularly since for generalized Rayleigh or Nakagami fading, we require nonintegral absolute moments. In general, we would need the probability density of the code cross-correlation; however, for any

TABLE 3

SINGLE-TERM APPROXIMATIONS FOR THE PROBABILITY OF ERROR FOR SPECIAL CASES OF GENERALIZED RAYLEIGH FADING FOR A SINGLE INTERFERER

RICIAN: $(m_1=m_2=1)$

$$[(1-\rho^2)/4]\ \exp(-\beta_1^2/\Omega_1)\ (P_2\Omega_2/P_1\Omega_1)\ \langle \hat{c}_n^2 \rangle \left(1 + \frac{\omega_2^2}{\Omega_1^2\Omega_2(1-\rho^2)}\right)$$

where $\omega_2 = \beta_2\ \Omega_1 - \rho\ \beta_1\ \Omega_1^{1/2}\ \Omega_2^{1/2}$

NAKAGAMI: $(m_1=m_2=m,\ \beta_1=\beta_2=0)$

$$\binom{2m-1}{m}^2 [(1-\rho^2)/4]^m\ (P_2\Omega_2/P_1\Omega_1)^m\ \langle |\hat{c}_n|^{2m} \rangle$$

RAYLEIGH: $(m_1=m_2=1,\ \beta_1=\beta_2=0)$

$$[(1-\rho^2)/4]\ (P_2\Omega_2/P_1\Omega_1)\ \langle \hat{c}_n^2 \rangle$$

INDEPENDENT FADING: $(\rho=0)$

$$\binom{2m_1-1}{m_1} \exp(-m_1\beta_1^2/\Omega_1)\ 2^{-m_1-m_2}\ (m_1P_2\Omega_2/m_2P_1\Omega_1)^{m_1}$$

$$L_{m_1}^{(m_2-1)}(-m_2\beta_2^2/\Omega_2)\ \langle |\hat{c}_n|^{2m_1} \rangle$$

Note: for Gold codes, $\langle \hat{c}_n^2 \rangle \approx 2/3L$ and $\langle |\hat{c}_n|^{2m} \rangle$ is approximately given by Eq.(2.22), after normalization.

reasonably long code, determination of the probability density is computationally impractical. Therefore, estimates of the moments are desired. For the case of Gold codes, an empirical estimate of the low-order even moments of the cross-correlation is given by Eq.(2.22). By straightforward extrapolation of this result to nonintegral absolute moments, we can readily evaluate the single- or two-term estimates of the average probability of error, provided the parameter m is not too large, i.e., provided the order of the moments required is not too large.

Figures 4 through 11 are plots of the average probability of error, the single-term estimates using the approximate moments given by Eq.(2.22), and the corresponding aproximation error for several examples of Rician and Nakagami fading. The curves are drawn for the case of equal power ($P_1\langle f_1^2\rangle = P_2\langle f_2^2\rangle$), equal fading statistics ($m_1=m_2$, $\beta_1=\beta_2$, $\Omega_1=\Omega_2$), and a subset of Gold codes of length 127. The plots of the exact values for the probability of error were obtained by evaluating the moments from the actual probability density for the code cross-correlation taken over a subset of 5 of the 129 Gold codes of length 127 (permitting 10 pairwise combinations out of a possible 8256 combinations). The probability density is determined by tabulating the relative frequency of cross-correlation values for all possible integer offsets ($\tau_i=nT_c$, $0\le n<L$) for each of the code pairs, for all possible data sequences on the interfering signal (b_i), and for all initial phases of the interfering signal's spreading code (circular shifts of the code within the data bit interval) [Chapter 2], [Hanlon, Peterson, and Gardner, 1980]. This was practicable due to the short code length and the limited code set. The plots show that the single-term estimates are quite good over a wide range of the channel fading parameters.

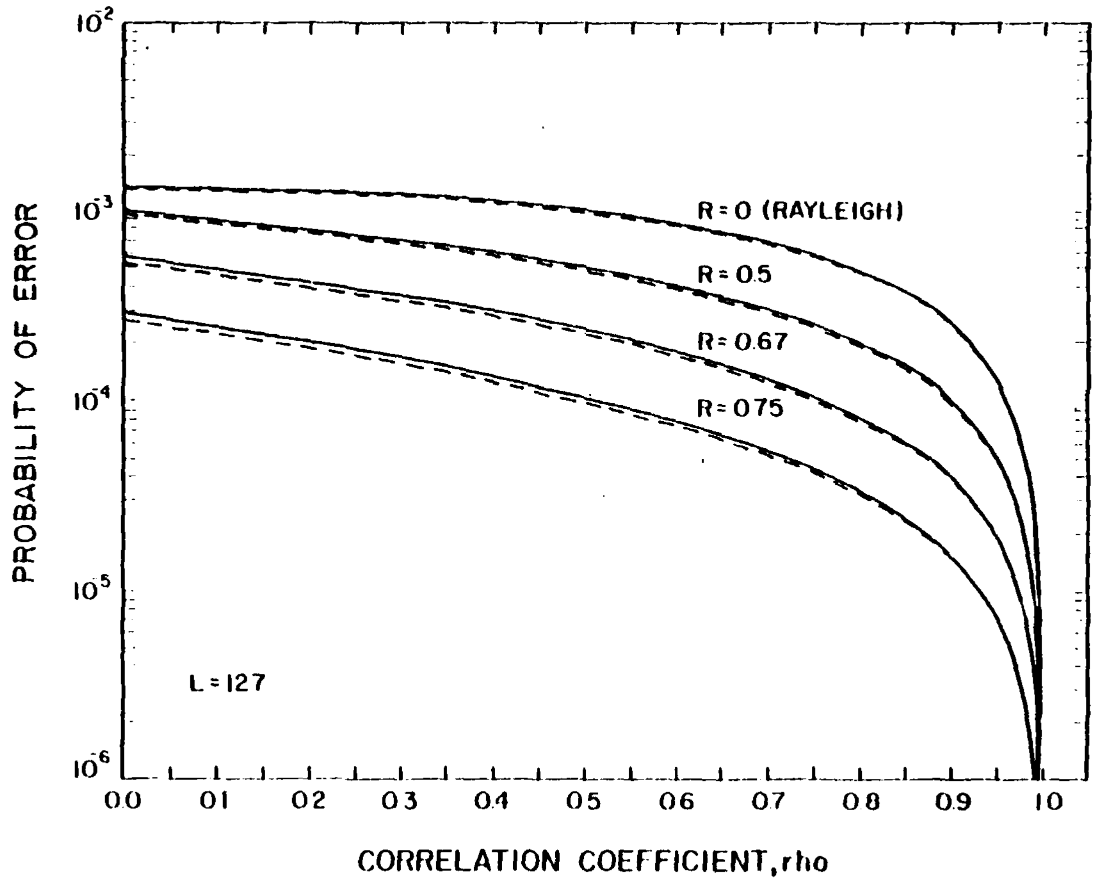

Figure 4. Probability of error versus correlation coefficient for Rician fading with varying specular power ratios. Exact series expansions (solid lines) and single-term approximations (dashed lines).

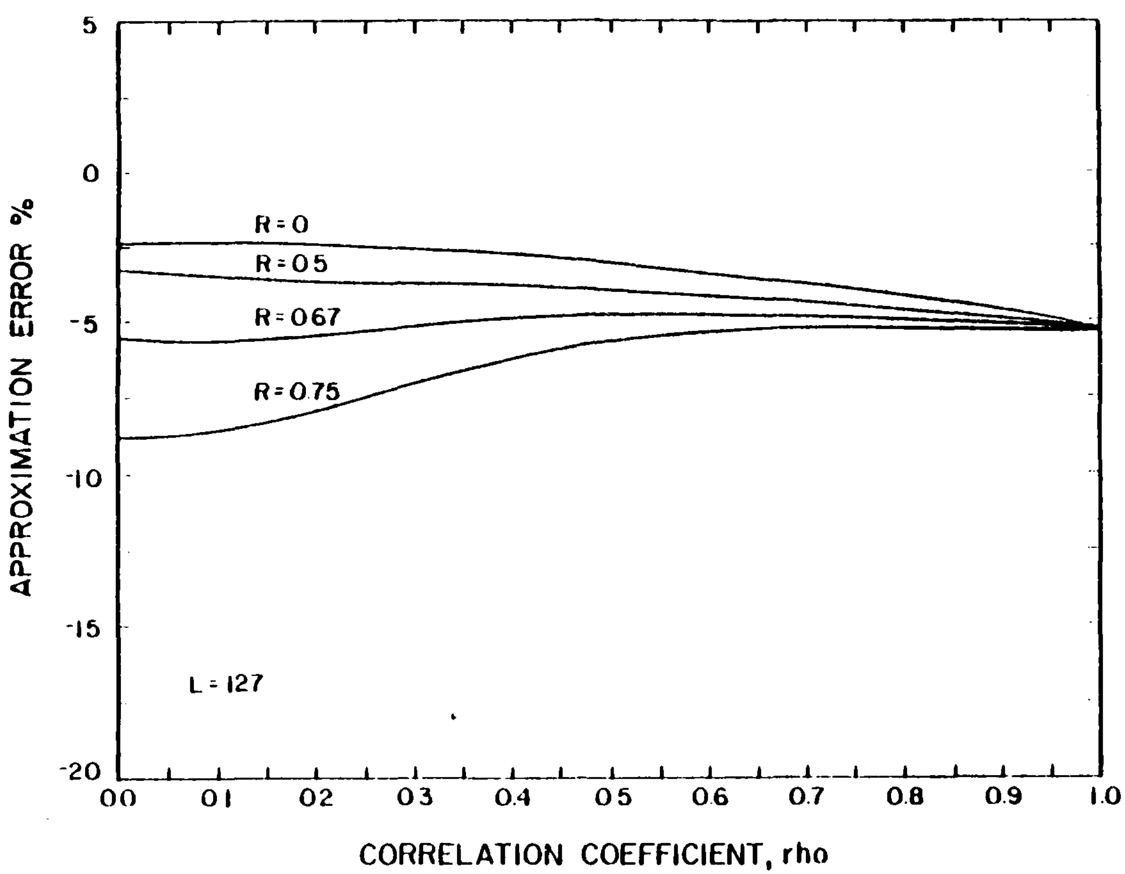

Figure 5. Single-term approximation error versus correlation coefficient for Rician fading with varying specular power ratios.

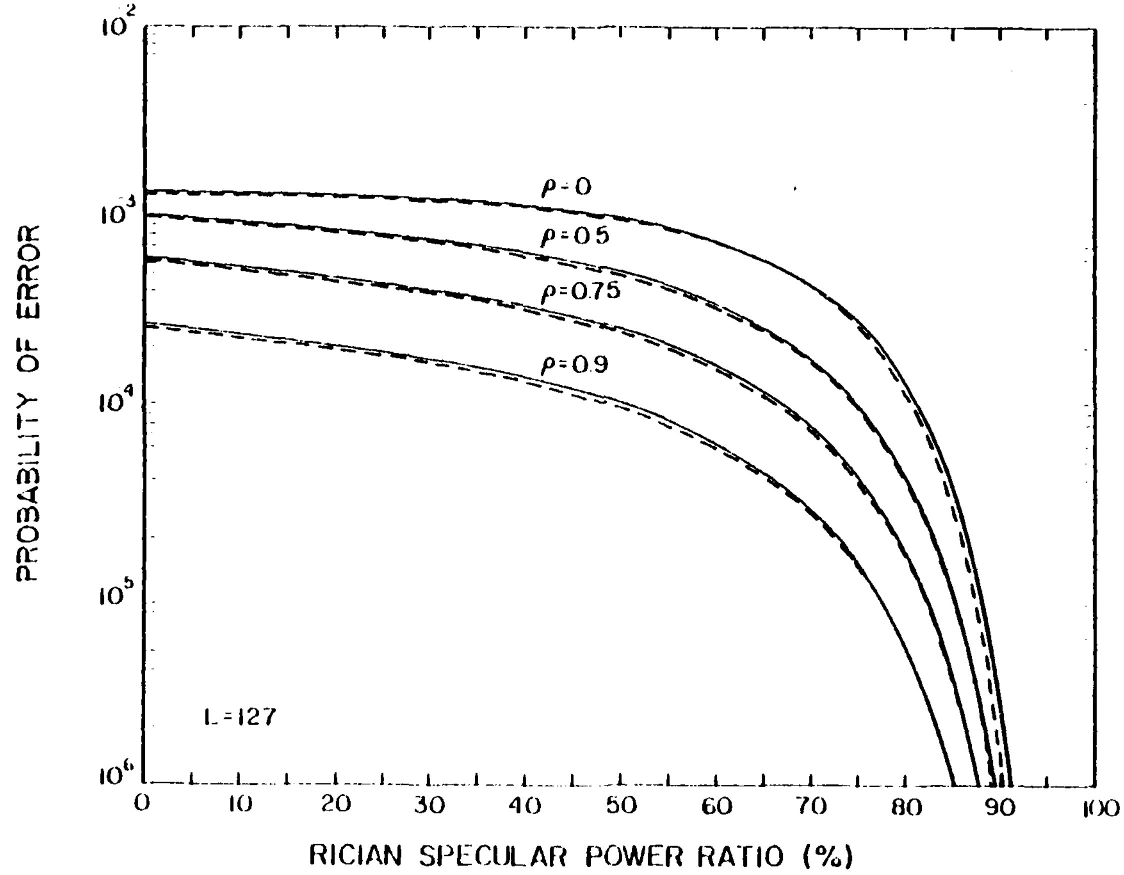

Figure 6. Probability of error versus Rician specular power ratio with varying correlation coefficients. Exact series expansions (solid lines) and single-term approximations (dashed lines).

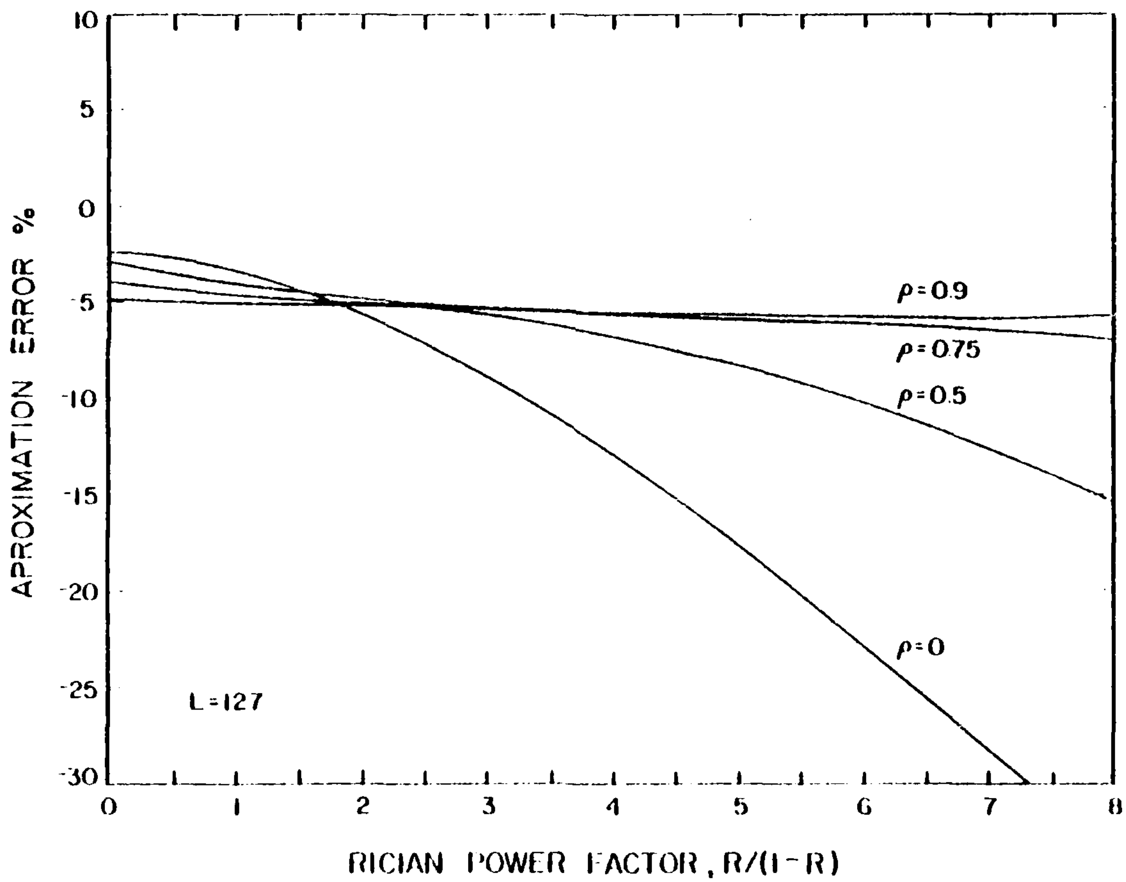

Figure 7. Single-term approximation error versus Rician power factor with varying correlation coefficients.

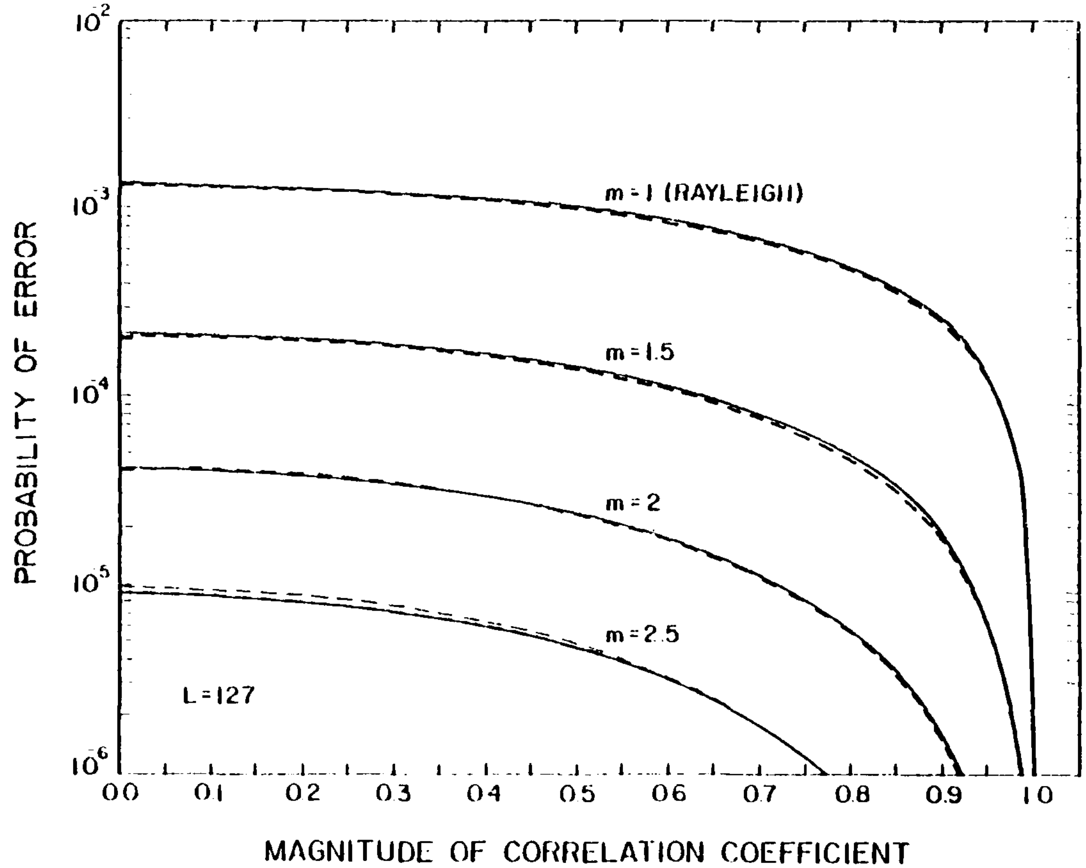

Figure 8. Probability of error versus magnitude of correlation coefficient for Nakagami fading with varying values of the m-distributed fading factor. Exact series expansions (solid lines) and single-term approximations (dashed lines).

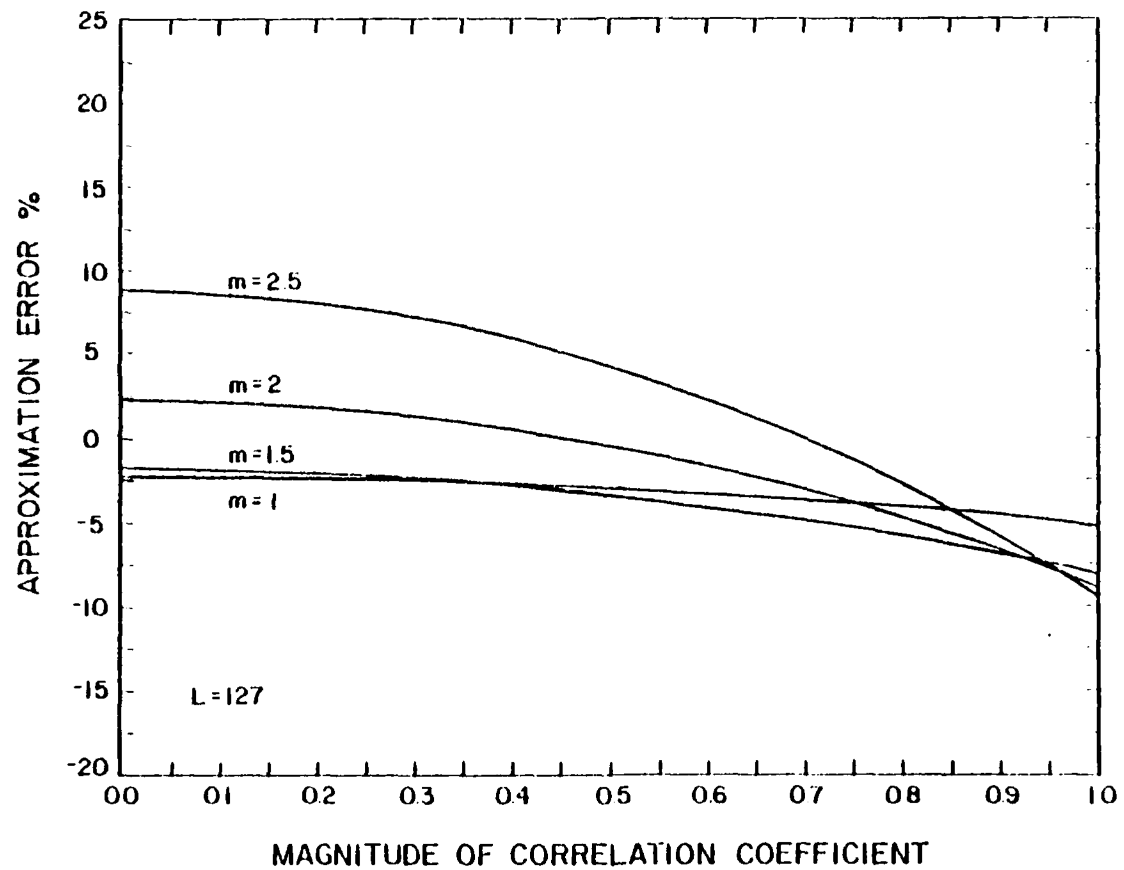

Figure 9. Single-term approximation error versus magnitude of correlation coefficient for Nakagami fading with varying values of the m-distributed fading factor.

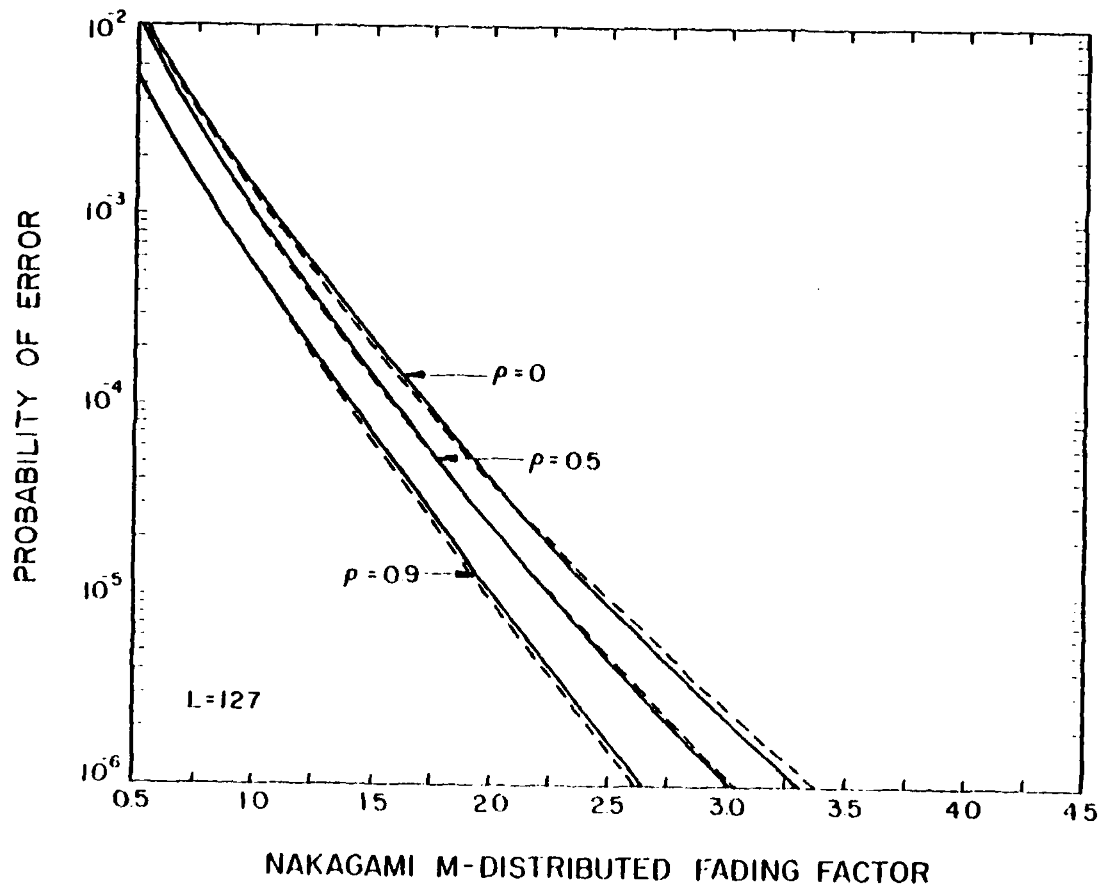

Figure 10. Probability of error versus Nakagami m-distributed fading factor *with varying correlation coefficients. Exact series expansions* (solid lines) and single-term approximations (dashed lines).

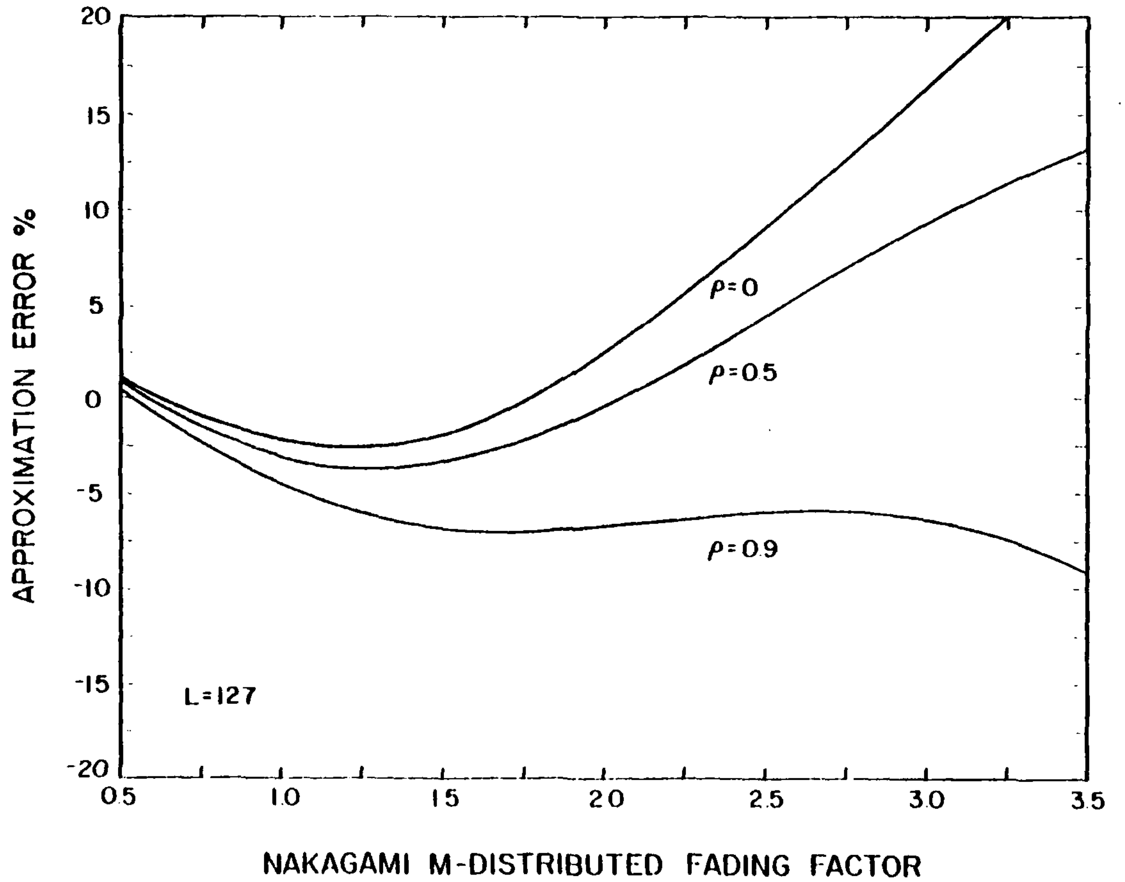

Figure 11. Single-term approximation error versus Nakagami m-distributed fading factor with varying correlation coefficients.

The Rician results are given for specular power ratios of 0, 50, 67, and 75 percent, which correspond to power factors [$R/(1-R) = \beta^2/\Omega$] of 0, 1, 2, and 3. As the correlation of the underlying Gaussians increases (ρ increases), the correlation between f_1 and f_2 increases and the probability of error drops towards zero. As the fading factors more closely agree statistically, the signal and the interference tend to maintain their relative strengths at the input to the receiver. As a result, it is less likely that the sum of the signal and the interference will result in a sign reversal in the detector and cause a bit error [Hanlon and Gardner, 1979]. This is also true for Nakagami fading and, in general, the import of the fading is significantly reduced for highly correlated channels. As the Rician specular power ratio increases (fading decreases), the probability of error decreases; however, the approximation error tends to increase. Use of more than one term of the series in the approximation would result in a wider range of the parameters over which close agreement is attained.

The Nakagami results are very similar to those for Rician fading. The error rate decreases as the fading decreases (m increases) and the approximation error tends to correspondingly increase. The empirical estimates of the code cross-correlation moments can be used to calculate additional terms of the series to improve the accuracy, provided m is not made larger than about 4. For larger values of m, the order of the moments required are too large to obtain good estimates of the moments from the empirical moments [Hanlon, Peterson, and Gardner, 1980].

The average probability of error for Rician fading is dominated by the depth of fading on the intended signal as evidenced by the exponential factor, $\exp(-\beta_1^2/\Omega_1)$. A similar dependence for Nakagami fading is reflected in the exponential dependence on m (order of the moments of the code cross-correlation). Since we have implicitly assumed that the m factor is the same for both fading intensities, the distinction is difficult to see. For the case of independent Nakagami fading, Eq.(3.6) clearly shows that the exponential dependence is on m_1 (fading on intended signal) rather than m_2 (fading on interfering signal).

While the performance for both the Rician and Nakagami fading cases demonstrates high dependence on the likelihood of deep fades on the intended signal, the performance shows significantly different dependence on the code length in the two cases. This is readily seen by comparing the single-term approximations. For Rician fading, the approximation is proportional to $\langle\hat{c}_n^2\rangle$ and, hence, from Eq.(2.22), L^{-1}, whereas, the single-term approximation for the Nakagami case is proportional to $\langle|\hat{c}_n|^{2m}\rangle$, or equivalently, L^{-m}. The basis for this difference can be seen by looking at the case of independent channels. For independent channels, the error rate can be alternatively represented as [Appendix C], [Hanlon and Gardner, 1979]

$$\Pr\{E\} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k!}\langle|\hat{c}_n|^k\rangle\,\langle|\cos\Theta|^k\rangle\,(P_2/P_1)^{k/2}\,\langle f_2^k\rangle\,p_{f_1}^{(k-1)}(0) \qquad (3.7)$$

where $\langle f_2^k\rangle$ is the k-th moment of the fading intensity on the interfering signal and $p_{f_1}^{(k-1)}(0)$ is the (k-1) derivative of the probability density for the fading

intensity of the intended signal evaluated at zero. This expression requires the derivatives at the origin to be finite, which, in general, is not the case for the generalized Rayleigh distribution. By restricting our interest to integral values of $(2m_1)$, we can see by expanding the probability density for f_1 in a power series that the first nonzero derivative at the origin is for k equal to $2m_1$. The larger the value of m_1 (the shallower the fading on the intended signal), the higher the order of the code cross-correlation moment which first contributes to the series. This is exactly the effect seen in Eq.(3.6), which can also be derived from Eq.(3.7) when $(2m_1)$ is an integer. From Eq.(2.22) we see that the first term varies as L^{-m}. For Rician fading, m is equal to one and the first term is always proportional to the normalized second moment, which is to say, inversely proportional to the code length. Consequently, the degree of improvement in performance to be expected from increasing the code length is highly dependent on the form of the distribution of the intended signal fading intensity, particularly the behaviour (derivatives) of the distribution near the origin (region of deep fades). As a result, we cannot, in general, approximate a given fading distribution with another distribution. For example, Nakagami gives the transformation for approximating a Rician distribution by a Nakagami distribution as [Nakagami, 1960], [Nakagami, et al., 1957]

$$\Omega_N = \Omega_R + \beta^2 \quad , \quad m = [1 - (\beta^4 / \Omega_N^2)]^{-1} \; . \tag{3.8}$$

This transformation is obtained by equating the second and fourth moments (Eq.(B.4)). In terms of the specular power ratio (Eq.(B.6)), this implies

$$m = (1 - R^2)^{-1} . \tag{3.9}$$

Under this transformation, specular power ratios of 0, 50, 67, and 75 percent correspond to m values of, respectively, 1, 1.33, 1.8, and 2.29. It is clear from Fig. 4 and Fig. 8 that this transformation does not provide comparable performance measures in terms of bit error rates. Likewise, any transformation which depends primarily on equating moments and, therefore, on the overall behaviour of the distribution, generally provides poor agreement since the error rate is determined primarily by the behaviour of the distribution in the region of deep fades [Barrow, 1962]. Regardless of the form of the distribution, relatively long code sequences are required to maintain low error rates unless the fading on the intended signal is very shallow or the fading correlation approaches unity.

3.2. Multiple Independent Interferers With Additive Noise

For the case of several simultaneous interferers, the individual interference terms in Eq.(1.9) are independent of each other due to the presence of their respective data bits and timing offsets. The probability density for each term will be symmetric due to the cross-correlation factors and phases. These factors also tend to scale the individual terms towards zero ($|c_n| \ll 1$, $|\cos \theta| \leq 1$), i.e., towards their means. Provided the P_i are on the same order (which will be assumed), then none of the individual terms dominates the series. Under these conditions, the composite interference at the output of the correlator (Z) will be asymptotically normal with zero mean and variance

$$\mathrm{Var}(Z) = \mathrm{Var}(N) + \mathrm{Var}(I) = (\eta_0 T_d/4) + (T_d^2/4)\langle\hat{c}_n^2\rangle \sum_{i=2}^{M} P_i \langle f_i^2\rangle \ . \tag{3.10}$$

The probability of error is (Eq.(1.12))

$$\Pr\{E\} = \frac{1}{2}\Pr\{|Z| > S\} \ . \tag{3.11}$$

Since Z is assumed to be zero-mean Gaussian, $|Z|$ is one-sided Gaussian, that is, generalized Rayleigh: $|Z| \sim R[1/2,0,\mathrm{Var}(Z)]$. Evaluation of the probability of error then reduces to determination of the probability that one generalized Rayleigh variate is greater than another when they are independent. From Eq.(B.16), we have

$$\Pr\{E\} = \frac{1}{2}\binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} \frac{(m_1+1/2)_k}{(m_1+1)_k} (-1)^k \, y^{k+m_1} \, L_k^{(m_1-1)}\left(\frac{m_1\beta_1^2}{\Omega_1}\right) \tag{3.12}$$

where, for M system users,

$$y = \frac{4m_1\mathrm{Var}(Z)}{P_1\Omega_1 T_d^2} = \frac{m_1}{\Omega_1}[2\ \mathrm{SNR}_G^{-1} + \alpha(M-1)\ \langle\hat{c}_n^2\rangle\ (\Omega_1 + \beta_1^2)] \tag{3.13}$$

$$\alpha = \frac{1}{M-1}\sum_{i=2}^{M} [P_i(\Omega_i + \beta_i^2)/P_1(\Omega_1 + \beta_1^2)] \tag{3.14}$$

and SNR_G is the signal-to-Gaussian-noise ratio defined by

$$\mathrm{SNR}_G = [P_1 T_d^2 \ / \ 2\ \mathrm{Var}(N)] = 2P_1 T_d / \eta_0 \ . \tag{3.15}$$

The usefulness of the series in Eq.(3.12) is restricted to values of y which are less than one. For values of y greater than one, the series does not converge. There are two approaches to circumvent this problem. First, we can evaluate the complement of the probability to obtain an expression for the probability of error when y is greater than one, that is,

$$\Pr\{E\} = [1 - \Pr\{S \geq |Z|\}] / 2$$

$$= [1 - y^{-1/2} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} (-y)^{-k} L_{k+1/2}^{(m_1-1)} (-m_1\beta_1^2/\Omega_1)] / 2 \,. \tag{3.16}$$

Alternatively, we can use the change of variables

$$t = y/(1 + y) \,. \tag{3.17}$$

By using the binomial theorem and Eq.(A.27), we then obtain

$$\Pr\{E\} = \frac{1}{2} \binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1) \sum_{n=0}^{\infty} \frac{(m_1)_n}{n!} t^{n+m_1}$$

$$\sum_{k=0}^{n} \frac{(m_1+1/2)_k(-n)_k}{(m_1)_k\,(m_1+1)_k} L_k^{(m_1-1)} (m_1\beta_1^2/\Omega_1) \,. \tag{3.18}$$

Since t is always less than one and less than y, the convergence properties of this expansion are much better. For small values of t, i.e., t much less than one, the average probability of error is very closely approximated by

$$\Pr\{E\} \approx \frac{1}{2}\, t^{m_1} \binom{m_1-1/2}{m_1} \exp(-m_1 \beta_1^2/\Omega_1) \ . \tag{3.19}$$

If the signal-to-Gaussian-noise ratio is made arbitrarily large, we have

$$\lim_{SNR_G \to \infty} y = \alpha m_1 \,(M-1)\, \langle \hat{c}_n^2 \rangle \,(1 + \beta_1^2/\Omega_1) \ . \tag{3.20}$$

For reasonably long code sequences, $\langle \hat{c}_n^2 \rangle$ is very small and y tends to be small. In this case, t and y are approximately equal and Eq.(3.19) very closely approximates the average probability of error. Since we are considering an arbitrarily large signal-to-Gaussian-noise ratio, this represents an irreducible error rate due to cochannel interference and fading. Table 4 lists the estimates of the irreducible error rates for the various special cases.

We have analyzed the multiple interferer case by assuming that the composite interference is asymptotically normal (Gaussian) with zero mean. As a result, the interference was specified by its variance (power). We would not typically expect the Gaussian assumption to hold if the number of interferers was very small (say less than five). This can be tested by looking at the single-term estimates for the irreducible error rate, evaluating them for a single interferer (M=2), and comparing the estimates with the results of Section 3.1 for a single interferer. The single-term estimate for the irreducible error rate for multiple users when M equals two is given by

$$\Pr\{E\} \approx \frac{1}{2}\binom{m_1-1/2}{m_1} \exp(-m_1 \beta_1^2/\Omega_1)\, [m_1 \langle \hat{c}_n^2 \rangle P_2(\Omega_2 + \beta_2^2)/P_1 \Omega_1]^{m_1} \ . \tag{3.21}$$

TABLE 4

SINGLE-TERM APPROXIMATIONS FOR THE IRREDUCIBLE ERROR RATE FOR MULTIPLE INDEPENDENT USERS

GENERALIZED RAYLEIGH: $\frac{1}{2} t^{m_1} \binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1)$

$$\simeq \frac{1}{2} \binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1)[\alpha m_1(M-1)\langle\hat{c}_n^2\rangle(1 + \beta_1^2/\Omega_1)]^{m_1}$$

where $\alpha = (M-1)^{-1} \sum_{i=2}^{M} [P_i(\Omega_i+\beta_i^2)/P_1(\Omega_1+\beta_1^2)]$

RICIAN: $t \exp(-\beta_1^2/\Omega_1) / 4 \simeq \alpha(M-1)\langle\hat{c}_n^2\rangle(1 + \beta_1^2/\Omega_1) \exp(-\beta_1^2/\Omega_1) / 4$
(m_1=1)

NAKAGAMI: $\frac{1}{2} t^{m_1} \binom{m_1-1/2}{m_1} \simeq \frac{1}{2} \binom{m_1-1/2}{m_1} [\alpha m_1(M-1)\langle\hat{c}_n^2\rangle]^{m_1}$
(β_1=0)

RAYLEIGH: $\frac{1}{2} \{1-[1+\alpha(M-1)\langle\hat{c}_n^2\rangle]^{-1/2}\} \simeq \frac{1}{4} \alpha(M-1)\langle\hat{c}_n^2\rangle$
(m_1=1, β_1=0)

Note: for Gold codes, $\langle\hat{c}_n^2\rangle \simeq 2/3L$.

Comparison with the single-term estimate for independent fading given in Table 3 indicates that there can be a significant difference. However, for Rician fading on both channels ($m_1=m_2=1$), the result given by Table 3 is identical to that given by Eq.(3.21) for Rician fading on the intended signal ($m_1=1$). This comparison extends to Rayleigh fading when β_1 is set equal to zero in the Rician estimates. This agreement arises because the single-term single-interferer estimates in both cases depend only on the interference power. In these cases, the Gaussian assumption for the interference is valid even when the number of interferers is minimal. However, comparison of the Nakagami results ($\beta_1=\beta_2=0$) does not provide the same agreement unless we look at the special case of Rayleigh fading. The difference arises because the Nakagami multiple user result is dependent on the m-th power of the interference power, that is, proportional to ($\langle\hat{c}_n^2\rangle^m$); whereas, the single interferer result (Section 3.1) depends on the (2m)-th absolute moment.

Although the expression for the probability of error does not appreciably simplify for the Rician case ($m_1=1$), for Nakagami fading statistics for the intended signal ($\beta_1=0$), Eq.(3.12), Eq.(3.16), and Eq.(3.18) reduce to

$$\Pr\{E\} = \frac{1}{2}\, t^{m_1} \binom{m_1-1/2}{m_1}\, {}_2F_1(1/2,m_1;m_1+1;t) = \frac{1}{2}\, I_t(m_1,1/2) \qquad (3.22)$$

where ${}_2F_1$ is a hypergeometric function and I_t is a normalized incomplete beta function. This expression is equivalent to a result given by Barrow [1962]. It is readily shown that

$$ {}_2F_1(1/2,m;m+1;t) = (1-t)^{-1/2} - \frac{t}{2(m+1)} \; {}_2F_1(3/2,m+1;m+2;t) \; . \qquad (3.23) $$

By truncating both hypergeometric series in this relation, we obtain the bounds

$$ \left[1 + \frac{mt}{2(m+1)} \right] \le {}_2F_1(1/2,m;m+1;t) \le \left[(1-t)^{-1/2} - \frac{t}{2(m+1)} \right] . \qquad (3.24) $$

Extension of these bounds to the probability of error for Nakagami fading is straightforward and provides the estimate (for small t):

$$ \Pr\{E\} \approx \frac{1}{4} \; t^{m_1} \begin{pmatrix} m_1-1/2 \\ m_1 \end{pmatrix} \left[1 + (1-t)^{-1/2} - \frac{t(m_1-1)}{2(m_1+1)} \right] . \qquad (3.25) $$

For the case of Rayleigh statistics ($m_1=1$, $\beta_1=0$), Eq.(3.22) reduces further to the well-known result:

$$ \Pr\{E\} = \frac{1}{2} \left[1 - (1-t)^{1/2}\right] = \frac{1}{2} \left[1 - (1+y)^{-1/2}\right] . \qquad (3.26) $$

Figures 12 and 13 are plots of the average probability of error versus the signal-to-Gaussian-noise ratio for the multiple user case for Rician and Nakagami (m-distributed) fading, respectively. Both sets of curves are drawn for equal effective power ($P_i = P_1$), mean square values normalized to one ($\langle f_i^2 \rangle = \langle f_1^2 \rangle = 1$), and $\langle \hat{c}_n^2 \rangle = 2/3L$. The figures clearly demonstrate that the average probability of error saturates at an irreducible error rate for

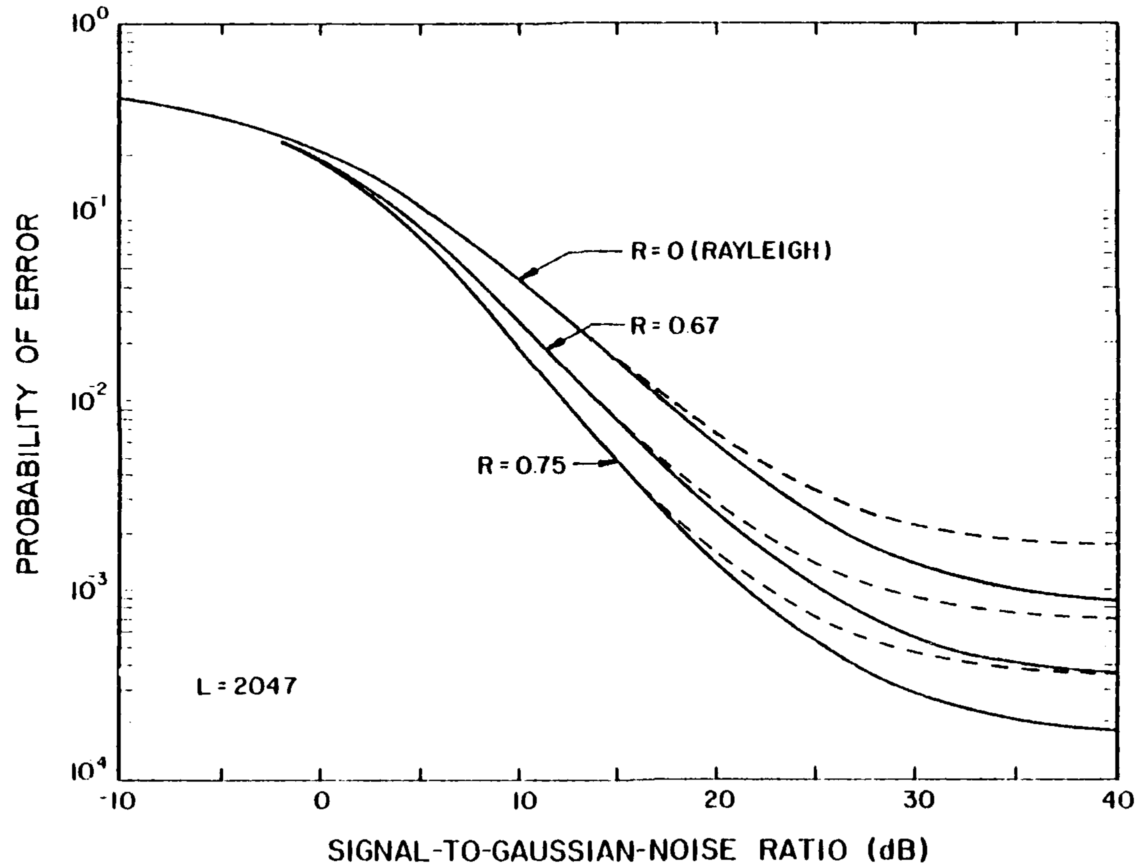

Figure 12. Probability of error versus signal-to-Gaussian-noise ratio for Rician fading with varying specular power ratios. Solid lines for 10 interferers and dashed lines for 20 interferers.

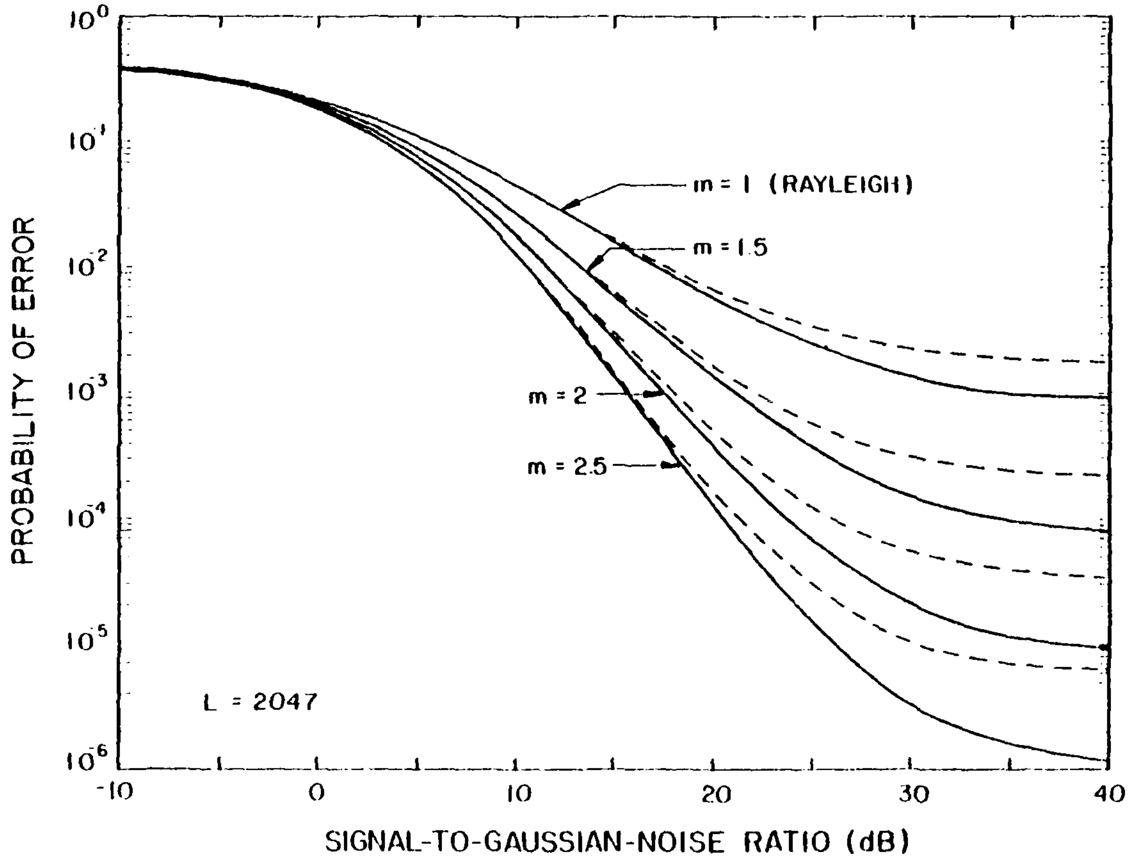

Figure 13. Probability of error versus signal-to-Gaussian-noise ratio for Nakagami fading with varying values of the m-distributed fading factor. Solid lines for 10 interferers and dashed lines for 20 interferers.

arbitrarily large signal-to-Gaussian-noise ratios. This effect is due to the combination of cochannel interference and fading on the intended signal. The deeper the fading on the intended signal or the greater the number of interferers, the more likely that the intended signal component falls below the level of the cochannel interference component in the receiver, thereby resulting in an error.

Chapter 4

SUMMARY, CONCLUSIONS, AND RECOMENDATIONS

4.1. Summary

This study has investigated the error performance of direct sequence spread spectrum multiple-access communications systems on nonselective fading channels. The receiver model used throughout the analysis is a coherent correlation receiver. The receiver correlates the received signal against a synchronized copy of the intended signal's spreading code sequence. The channel model assumes slow, nonselective fading where the fading intensities are distributed as generalized Rayleigh variates. Results are obtained for the special cases of a single interferer without additive noise and multiple independent interferers with additive noise. The results are developed as a series expansion in the code cross-correlation moments (single interferer) or as a series expansion in the powers of the code cross-correlation second moment (multiple interferers). To develop simple estimates of the probability of error, empirical estimates of the low-order cross-correlation moments of the spreading codes (taken to be Gold codes) are developed from a gamma density approximation based on the cross-correlation moments for both Gold codes and random codes. Numerical results for the probability of error are obtained with these estimates and compared with the results based on exact moments developed from the code cross-correlation probability density.

4.2. Conclusions

The analysis developed in this paper has shown that the method of series expansion in the code cross-correlation moments provides a tractable solution to evaluation of the error performance of direct sequence multiple-access communications systems operating over nonselective fading channels. In particular, truncation of the series provides simple single-term estimates which are accurate over a wide range of the channel fading parameters. These results show that the error rate is highly dependent on the form of the intended signal's fading distribution in the region of deep fades. Consequently, approximation of a given distribution by another distribution (e.g., approximation of a Rician distribution with a Nakagami distribution) is generally not feasible unless great care is taken to match the distributions in the region of deep fades. As a result, the channel must be carefully modeled. Use of the generalized Rayleigh distribution has encompassed the results for a wide class of fading distributions.

Use of empirical estimates of the cross-correlation moments has been shown to provide highly accurate estimates of the error rate unless the fading is very shallow. In addition, it has been shown that the direct use of moments based on integral timing offsets between the codes does not generally provide good estimates. The moments must be calculated for arbitrary timing offsets; however, the moments for arbitrary offset can be readily developed from the moments for integral timing offsets.

It has also been shown that the effect of fading on the error rate is significantly reduced for highly correlated channels. Further, unless the fading correlation is either close to unity or the fading on the intended signal is very shallow, long spreading codes will be required to maintain low error rates.

4.3. Recommendations

The analysis presented here has been restricted to the case of slow, nonselective fading. With this basis, extension of the results to relatively fast, nonselective fading would provide additional insight into direct sequence system performance over generalized Rayleigh fading channels. Alternatively, the effect of time variations in the channel characterization could be explored, that is, the channel fading parameters could be taken as random variables with some assumed distribution. The analysis would proceed as presented and then the expectation over the fading parameter distributions would provide the long-term average error rate. In general, this extended analysis may not prove tractable. However, the single-term estimates of the average probability of error could be analyzed in this fashion with reasonable anticipation of success.

An alternate area of investigation would be the analysis of diversity gain (e.g., space diversity) possible with direct sequence systems operating over generalized Rayleigh channels. This analysis would be particularly important for channels whose fading depths were on the order of Rayleigh fading or deeper.

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APPENDIX A

MATHEMATICAL FUNCTIONS AND FORMULAS

To simplify the references in this appendix, they are abbreviated to two letters designating the work, followed by the section, page or equation number in the reference. The two-letter abbreviations correspond as follows: AB-[Abramowitz, 1964], ER-[Erdelyi, 1953], GR-[Gradshteyn and Ryshik, 1965], HN-[Hansen, 1975], MO-[Magnus, Oberhettinger, and Soni, 1966] RA-[Rainville, 1960], VT-[Van Trees, 1968].

1. Pochhammer's symbol [AB6.1.22, ER1.21(5), GRp.xliii, HN2.4, RAs.18]

$$(a)_k = a(a+1)(a+2)\ldots(a+k-1) = \Gamma(a+k) \;/\; \Gamma(a) \;, \quad (a)_0 = 1 \tag{A.1}$$

2. Hypergeometric function [AB15.1.1, ER2.1.1(2), GR9.100, HN10.9.2, RAs.29(1)]

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \, k!} \, x^k \tag{A.2}$$

3. Linear (Euler) transformation for hypergeometric function [AB15.3.3,4; ER2.1.4(22),(23); GR9.131(1); HN10.9.2; MOp.47; RAs.38(4),(5)]

$$\begin{aligned} {}_2F_1(a,b;c;x) &= (1-x)^{c-a-b} \; {}_2F_1(c-a,c-b;c;x) \\ &= (1-x)^{-a} \; {}_2F_1(a,c-b;c;\tfrac{x}{x-1}) \end{aligned} \tag{A.3}$$

4. Hypergeometric function with unit argument [AB15.1.20, ER2.1.3(14), GR9.122, HN7.4.14, MOp.40, RAs.32]

$$ {}_2F_1(a,b;c;1) = [\Gamma(c)\ \Gamma(c-a-b)]\ /\ [\Gamma(c-a)\ \Gamma(c-b)] \tag{A.4} $$

where $c \neq 0, -1, -2, \ldots$; $\mathrm{Re}\,(c-a-b) > 0$.

5. Confluent (degenerate) hypergeometric function [AB13.1.2, ER6.1(1), GR9.210(1), HN10.8.3, MOp.262, RAs.68(1)]

$$ {}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\ k!}\ x^k \tag{A.5} $$

6. Kummer's transformation for the confluent hypergeometric function [AB13.1.27, ER6.3(7), GR9.212(1), HN10.8.3, MOp.267, RAs.69(2)]

$$ {}_1F_1(a;b;x) = e^x\ {}_1F_1(b-a;b;-x) \tag{A.6} $$

7. Generalized hypergeometric function [ER4.1(1), GR9.14(1), HN10.48.1, MOp.62, RAs.44(2)]

$$ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k\ k!}\ x^k \tag{A.7} $$

8. Confluent hypergeometric function of three variables [HN48.21.18]

$$ \Phi_3(a,b;c;x,y,z) = \sum_{k,n,m=0}^{\infty} \frac{(a)_k\ (b)_n}{k!\ n!\ m!\ (c)_{k+n+m}}\ x^k\ y^n\ z^m \tag{A.8} $$

9. Binomial function (binomial theorem) [AB15.1.8, ER2.8(4), HN10.4.6, MOp.38, RAs.45(3)]

$$(1 - x)^{-a} = {}_1F_0(a;;x) = {}_2F_1(a,b;b;x) \quad \text{(A.9)}$$

10. Binomial coefficient [AB6.1.21, ER1.21(6), MOp.4]

$$\binom{x}{y} = \Gamma(x+1) \,/\, [\Gamma(y+1)\ \Gamma(x-y+1)] \ , \binom{x}{n} = (-1)^n\ (-x)_n \,/\, n! \quad \text{(A.10)}$$

11. Generalized Laguerre function [ER6.9.2(37), MOp.336]

$$L_r^{(a)}(x) = \binom{r+a}{r}\ {}_1F_1(-r;a+1;x) \quad , \quad L_r^{(0)}(x) = L_r(x) \quad \text{(A.11)}$$

When r is a nonnegative integer, the Laguerre functions are Laguerre polynomials [AB13.6.9, ER6.9.2(36), GR8.972(1), HN10.8.4, RAs.112(1)]. For example,

$$L_0^{(a)}(x) = 1, \quad L_1^{(a)}(x) = a+1-x, \quad L_2^{(a)}(x) = \tfrac{1}{2}[x^2-2x(a+2)+(a+1)(a+2)] \quad \text{(A.12)}$$

12. Gamma function (Euler's integral of the second kind) [AB6.1; ER1.1; GR8.31; MOp.1; RAs.8,s.12]

$$\Gamma(x+1) = \int_0^\infty dt\ \ e^{-t}\ \ t^x \ , \quad \mathrm{Re}\ x > 0 \ ; \quad \Gamma(x+1) = x\ \Gamma(x) = x! \quad \text{(A.13)}$$

13. Legendre's duplication formula [AB6.1.18; ER1.2(15); GR8.335(1); HN2.6,2.7; MOp.3, RAs.19]

$$\Gamma(2x) = 2^{2x-1}\ \Gamma(x)\Gamma(x+1/2)/\Gamma(1/2), \quad (2x)_{2k} = 2^{2k}\ (x)_k\ (x+1/2)_k \tag{A.14}$$

14. Incomplete gamma function [AB6.5, ER9.1, GR8.35, HN6.10.1, MOp.337]

$$\Gamma(a,x) = \int_x^\infty dt\ \ e^{-t}\ \ t^{a-1} \ = \ \Gamma(a) - a^{-1}\ x^a\ \ {}_1F_1(a;a+1;-x) \tag{A.15}$$

The integral representation requires $|\arg x| < \pi$.

15. Beta function (Euler's integral of the first kind) [AB6.2, ER1.5, GR8.38, MOp.7, RAs.16]

$$B(a,b) = \Gamma(a)\ \Gamma(b)\ /\ \Gamma(a+b) = \int_0^1 dt\ \ t^{a-1}\ (1-t)^{b-1}$$

$$= 2\int_0^{\pi/2} dt\ (\sin t)^{2a-1}\ (\cos t)^{2b-1} \tag{A.16}$$

The integral representations require Re $a > 0$, Re $b > 0$.

16. Incomplete beta function [AB6.6, ER2.5.3, GR8.391, HN6.11.12, MOp.356]

$$B_x(a,b) = \int_0^x dt\ \ t^{a-1}\ (1-t)^{b-1} = a^{-1}\ x^a\ \ {}_2F_1(a,1-b;a+1;x) \tag{A.17}$$

17. Normalized incomplete beta function [AB6.6,26.5; ER2.5.3, GR8.392]

$$I_x(a,b) = B_x(a,b) \ / \ B(a,b) \ = \ 1 - I_{1-x}(b,a) \qquad (A.18)$$

18. Error function and complementary error function [AB7.1, ER6.9.2(23), GR8.25, HNp.507, MOp.349, RAp.127]

$$\mathrm{erf}(x) = [2/\Gamma(1/2)] \int_0^x dt \ e^{-t^2} = 1 - \mathrm{erfc}(x) = 2 \ P(2^{1/2}x) - 1$$

$$= 2x \ {}_1F_1(1/2;3/2;-x^2) \ / \ \Gamma(1/2) \qquad (A.19)$$

where P is the standard normal cumulative distribution function.

19. Modified Bessel function of the first kind (order r) [AB9.6.47, ER7.2.2(12), GR8.445, HN10.7.11, MOp.66, RAs.65(1)]

$$I_r(2x) = x^r \ {}_0F_1(;r+1;x^2) \ / \ \Gamma(r+1) \qquad (A.20)$$

20. Hille-Hardy formula (bilinear generating function for Laguerre polynomials) [ER10.12(20), GR8.976(1), HN48.21.20, MOp.242, RAs.120]

$$\sum_{k=0}^{\infty} \frac{k!}{(a+1)_k} \ t^k \ L_k^{(a)}(x) \ L_k^{(a)}(y) = (1-t)^{-1} \ \Gamma(a+1) \ (xyt)^{-a/2}$$

$$\exp[t(x+y)/(t-1)] \ \ I_a[2(xyt)^{1/2}/(1-t)] \ , \ \ |t| < 1 \qquad (A.21)$$

21. Bilinear generating function for Laguerre polynomials [HN48.21.18]

$$\sum_{k=0}^{\infty} \frac{(m+1)_k}{(c+m+1)_k} t^k L_k^{(a)}(x) L_{k+m}^{(c)}(y) = \frac{(c+1)_m}{m!} (1-t)^{-a-1}$$

$$\exp[t(x+y)/(t-1)]\ \Phi_3[-m,c-a+m;c+1;y/(1-t),yt/(1-t),xyt(1-t)^{-2}] \tag{A.22}$$

For m equal to zero and c equal to a, this reduces to the Hille-Hardy formula.

22. Laguerre series for modified Bessel functions of the first kind [AB22.9.16, ER10.12(18), GR8.975(3), HN48.7.2, MOp.242, RAs.113(2)]

$$\sum_{k=0}^{\infty} \frac{1}{(1+a)_k} (-1)^k\ t^{2k}\ L_k^{(a)}(x^2) = \Gamma(a+1)\ (xt)^{-a}\ e^{-t^2}\ I_a(2xt) \tag{A.23}$$

23. Generalized Marcum's Q-function [VTp.411]

$$Q_m(a,b) = \int_b^{\infty} dx\ \ a\ (x/a)^m\ \exp[-\tfrac{1}{2}(x^2 + a^2)]\ I_{m-1}(ax) \tag{A.24}$$

$$Q_1(a,b) = Q(a,b),\ Q_m(a,0) = 1,\ Q_m(0,b) = \Gamma(m,b^2/2)/\Gamma(m) \tag{A.25}$$

24. Marcum's Q-function [HN58.5.3, VTp.395]

$$Q(a,b) = \exp[-\tfrac{1}{2}(a^2 + b^2)] \sum_{k=0}^{\infty} (a/b)^k\ I_k(ab)\quad ,\quad a \le b$$

$$= 1 - \exp[-\tfrac{1}{2}(a^2 + b^2)] \sum_{k=1}^{\infty} (b/a)^k\ I_k(ab)\quad ,\quad b \le a \tag{A.26}$$

25. Integral of Bessel function, exponential and power [AB11.4.28, ER7.7.3(22), GR6.631(1)]

$$\int_0^\infty dx\ \exp(-a^2x^2)\ x^{s-1}\ I_r(2bx) = \frac{(b/a)^r\ \Gamma(\frac{r+s}{2})}{2a^s\ \Gamma(r+1)}\ {}_1F_1[(r+s)/2;r+1;(b/a)^2]$$

$$= \frac{1}{2}\Gamma[(s-r)/2]\ (b/a)^r\ a^{-s}\ \exp(b^2/a^2)\ \ L^{(r)}_{((s-r)/2)-1}(-b^2/a^2) \qquad \text{(A.27)}$$

where Re $a^2 > 0$, Re $(s+r) > 0$.

26. Interchange of order of summation [HNs.4.1, RAs.37]

$$\sum_{n=0}^\infty \sum_{k=0}^\infty f(k,n) = \sum_{k=0}^\infty \sum_{n=0}^\infty f(k,n) = \sum_{n=0}^\infty \sum_{k=0}^n f(k,n-k)$$

$$= \sum_{n=0}^\infty \sum_{k=0}^n f(n-k,k) \qquad \text{(A.28)}$$

27. Powers of series [AB3.6.20, GR0.314]

$$\left(\sum_{k=0}^\infty a_k\ x^k\ \right)^n = \sum_{k=0}^\infty c_k\ x^k\ , \qquad \text{(A.29)}$$

where

$$c_0 = a_0^n\ ,\quad c_m = \frac{1}{ma_0}\sum_{k=1}^m\ (kn-m+k)\ a_k\ c_{m-k}\quad \text{for } m\geq 1 \qquad \text{(A.30)}$$

28. Double factorial function [ABp.258, GRp.xliii]

$$(2n)!! = (2)(4)\ldots(2n) = 2^n\, n! \quad , \; (2n-1)!! = (1)(3)\ldots(2n-1) \qquad (A.31)$$

29. Generating function for Stirling numbers of the first kind [AB24.1.3, HN52.2.1]

$$(-x)_n = (-1)^n \sum_{m=0}^{n} x^m\; S_1(n,m) \qquad (A.32)$$

30. Recurrence relation for Stirling numbers of the first kind [AB24.1.3]

$$S_1(n+1,m) = S_1(n,m-1) - nS_1(n,m) \; , \; n \geq m \geq 1 \; ; \quad S_1(n,0) = \delta_{0n}, \qquad (A.33)$$

$$S_1(n,1) = (-1)^{n-1}\,(n-1)!, \; S_1(n,n) = 1, \; S_1(n,n-1) = -n(n-1)/2 \qquad (A.34)$$

31. Generating function for Stirling numbers of the second kind [AB24.1.4, HN52.2.15]

$$(-x)^n = \sum_{m=0}^{n} (-1)^m\, (x)_m \;\; S_2(n,m) \qquad (A.35)$$

32. Recurrence relation for Stirling numbers of the second kind [AB24.1.4]

$$S_2(n+1,m) = mS_2(n,m) + S_2(n,m-1) \; , \; n \geq m \geq 1 \; ; \qquad (A.36)$$

$$S_2(n,0) = \delta_{0n}, \; S_2(n,1) = S_2(n,n) = 1, \; S_2(n,n-1) = n(n-1)/2 \qquad (A.37)$$

33. Series involving Stirling numbers of the second kind [HN6.7.22]

$$\sum_{k=1}^{n} (-1)^k (-n)_k k^m / k! = 2^n \sum_{k=1}^{m} (-n)_k (-2)^{-k} S_2(m,k) \tag{A.38}$$

34. Generating function for generalized Euler numbers of order (-m) [HN51.7, MOp.32]

$$(\cos t)^m = \sum_{k=0}^{\infty} (-1)^k t^{2k} E_{2k}^{(-m)} / (2k)! \tag{A.39}$$

From Hansen [HN88.1.2], we have

$$(\cos t)^m = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k} [2^{-m} \sum_{n=0}^{m} \binom{m}{n} (2n-m)^{2k}] . \tag{A.40}$$

By comparing Eq.(A.40) and Eq.(A.39), we obtain [Hanlon, Peterson, and Gardner, 1980]

$$E_{2k}^{(-m)} = 2^{-m} \sum_{n=0}^{m} \binom{m}{n} (2n-m)^{2k} . \tag{A.41}$$

35. Generating function for Bernoulli numbers [AB23.1.1, ER1.13(1), GR9.610, HN50.5.3, RAs.153(8)]

$$t / (e^t - 1) = \sum_{n=0}^{\infty} B_n t^n / n! \quad , \quad |t|<2\pi \tag{A.42}$$

For example,

$$B_0 = 1 \; , \quad B_1 = -1/2 \; , \quad B_2 = 1/6 \; , \quad B_4 = -1/30 \tag{A.43}$$

36. Series expansion for log(cos x) in Bernoulli numbers [AB4.3.72, ER1.20(6), GR1.518(2), HN50.6.8]

$$\log(\cos x) = \sum_{n=1}^{\infty} (-1)^n \, 2^{2n-1} \, (2^{2n}-1) \, B_{2n} \, x^{2n}/[n(2n)!], \quad |x|<\pi/2 \tag{A.44}$$

APPENDIX B

PROBABILITY DENSITY FOR FADING INTENSITY

B.1. Univariate Density

Let $y_1,\ldots,y_n$ be independent and identically distributed normal random variables with mean μ and variance σ^2. The Euclidean norm of these variables, designated f, is defined by

$$f = \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2} . \tag{B.1}$$

It is well-known that f has a noncentral chi distribution with n degrees of freedom and its probability density is given by [Miller, Bernstein, and Blumenson, 1958],[Nakagami, 1960]

$$p_f(x) = \frac{2mx^m}{\Omega\beta^{m-1}} \exp\{-m(x^2 + \beta^2)/\Omega\} \, I_{m-1}(2m\beta x/\Omega) \quad , \quad x \geq 0 \tag{B.2}$$

where I_r is a modified Bessel function of the first kind of order r and

$$m = n/2 \quad , \qquad \Omega = n\,\sigma^2 \quad , \qquad \beta = n^{1/2}\,\mu \quad . \tag{B.3}$$

Although this development requires n to be a positive integer, Eq.(B.2) is a valid fading probability density for all m not less than 1/2. In this context, f is referred to as a generalized Rayleigh variate (generalized n-distributed variate) [Nakagami, 1960],[Miller, Bernstein, and Blumenson, 1958]. For a

random variable whose probability density is given by Eq.(B.2), its distribution is represented by $f \sim R(m,\beta,\Omega)$. The moments of f are readily evaluated by using Eq.(A.27) and are given by [Nakagami, 1960],[Krishnan, 1967]

$$\langle f^{\nu}\rangle = \Gamma(m + \nu/2)\ (\Omega/m)^{\nu/2}\ {}_1F_1(-\nu/2;m;-m\beta^2/\Omega)\ /\ \Gamma(m)$$

$$= \Gamma(\nu/2 + 1)\ (\Omega/m)^{\nu/2}\ L_{\nu/2}^{(m-1)}(-m\beta^2/\Omega)\ ,\ \nu+2m > -1 \tag{B.4}$$

where $\langle\ \rangle$ represents expectation, ${}_1F_1$ is a confluent hypergeometric function, and $L_r^{(a)}$ is a generalized Laguerre function. Note that the mean square value of f is just

$$\langle f^2\rangle = \Omega + \beta^2\ . \tag{B.5}$$

The generalized Rayleigh distribution encompasses a wide class of distributions which are useful in the study of communications over fading channels. For m equal to one and β equal to zero, f has a Rayleigh distribution. For m equal to one and β arbitrary, f has a Rician distribution (Nakagami-Rice or n-distribution). In this case, β^2 is the specular power and Ω is the mean square value of the Rayleigh component. The depth of fading can be measured by the specular power ratio (the ratio of the specular power to the mean square value), designated R and defined by

$$R = \beta^2 / (\Omega + \beta^2) . \tag{B.6}$$

The depth of fading increases with decreasing R. However, Rician fading can never be deeper than Rayleigh fading ($\beta = 0$). For β equal to zero and m not less than 1/2, f has a Nakagami distribution (m-distributed). In this case, the mean square value of f is Ω and the variance of f^2 is (Ω^2/m). Hence, m is the inverse of the normalized variance of f^2 [Nakagami, 1960]. The depth of fading increases with decreasing m. For values of m less than one, the fading is more severe than Rayleigh fading, and for values of m greater than one, is shallower than Rayleigh fading. Consequently, Rayleigh fading is a special case of both Rician and Nakagami fading. In the general case, the fading decreases as either m or β increases.

The cumulative distribution function for a generalized Rayleigh variate is given by

$$F_f(x) = 1 - Q_m[(2m\beta^2/\Omega)^{1/2} , (2mx^2/\Omega)^{1/2}] \tag{B.7}$$

where Q_m is a generalized Marcum's Q-function. To obtain an alternate expression, we make use of the Laguerre series for modified Bessel functions (Eq.(A.23)) to express the probability density as

$$p_f(x) = \frac{2mx}{\Omega\Gamma(m)} \exp(-m\beta^2/\Omega) \sum_{k=0}^{\infty} \frac{(-1)^k}{(m)_k} \left(\frac{mx^2}{\Omega}\right)^{m+k-1} L_k^{(m-1)}(m\beta^2/\Omega) . \tag{B.8}$$

Integration of the probability density is then straightforward and provides

[Hanlon and Gardner, 1980]

$$F_f(x) = \frac{1}{\Gamma(m+1)} \exp(-m\beta^2/\Omega) \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+1)_k} \left(\frac{mx^2}{\Omega}\right)^{m+k} L_k^{(m-1)}(m\beta^2/\Omega) . \qquad (B.9)$$

The probability that one generalized Rayleigh variate exceeds another when they are independent is given by

$$\Pr\{f_2 > f_1\} = \int_0^{\infty} dx\ p_{f_2}(x) \int_0^{x} dy\ p_{f_1}(y)$$

$$= \int_0^{\infty} dx\ p_{f_2}(x)\ F_{f_1}(x) = \langle F_{f_1}(f_2)\rangle . \qquad (B.10)$$

By using Eq.(B.9) and Eq.(B.4), we readily obtain

$$\Pr\{f_2 > f_1\} = \exp(-m_1\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} (-1)^k (m_1\Omega_2/m_2\Omega_1)^{m_1+k}$$

$$L_k^{(m_1-1)}(m_1\beta_1^2/\Omega_1)\ L_{k+m_1}^{(m_2-1)}(-m_2\beta_2^2/\Omega_2) . \qquad (B.11)$$

Note that if the ratio $(m_1\Omega_2/m_2\Omega_1)$ is greater than one, the series does not converge and the representation given by Eq.(B.11) is formal. In this case, we can use the alternate expression

$$\Pr\{f_2 > f_1\} = 1 - \Pr\{f_1 \geq f_2\} \qquad (B.12)$$

which can be written from Eq.(B.11) by inspection. Several special cases of the general result are of interest:

(a.) For f_1 and f_2 both Rician ($m_1=m_2=1$), use of Eq.(A.22), Eq.(A.20), and Eq.(A.26) provides [Schwartz, Bennett, and Stein, 1966], [Van Trees, 1968]

$$\Pr\{f_2 > f_1\} = Q([2\beta_2^2/(\Omega_1+\Omega_2)]^{1/2}\ ,\ [2\beta_1^2/(\Omega_1+\Omega_2)]^{1/2})$$

$$- [\Omega_1/(\Omega_1+\Omega_2)]\ \exp[-(\beta_1^2+\beta_2^2)/(\Omega_1+\Omega_2)]\ I_0[2\beta_1\beta_2/(\Omega_1+\Omega_2)]\ . \tag{B.13}$$

(b.) For f_1 and f_2 both Nakagami m-distributed ($\beta_1=\beta_2=0$), Eq.(B.11) readily reduces to

$$\Pr\{f_2 > f_1\} = \begin{pmatrix} m_1+m_2-1 \\ m_1 \end{pmatrix} \left(\frac{m_1\Omega_2}{m_2\Omega_1}\right)^{m_1} {}_2F_1(m_1, m_1+m_2; m_1+1; -m_1\Omega_2/m_2\Omega_1)\ . \tag{B.14}$$

After linear transformation of the hypergeometric function and use of Eq.(A.17) and Eq.(A.18), we obtain

$$\Pr\{f_2 > f_1\} = I_t(m_1,m_2)\quad ,\quad t = [m_1\Omega_2\ /\ (m_1\Omega_2 + m_2\Omega_1)]\ . \tag{B.15}$$

(c.) For f_2 one-sided Gaussian (m_2 = 1/2, β_2 = 0), we obtain by direct substitution,

$$\Pr\{f_2 > f_1\} = \binom{m_1-1/2}{m_1} \exp(-m_1\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} \frac{(m_1+1/2)_k}{(m_1+1)_k} (-1)^k$$

$$\left(\frac{2m_1\Omega_2}{\Omega_1}\right)^{m_1+k} L_k^{(m_1-1)}(m_1\beta_1^2/\Omega_1) \ . \tag{B.16}$$

Table B.1 lists the results for several additional special cases.

B.2. Bivariate Density

Let the pairs $(y_{11},y_{12}),\ldots,(y_{n1},y_{n2})$ be independent and identically distributed where the elements y_{j1} and y_{j2} are jointly normal for all j with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation coefficient ρ. Then the norms, f_1 and f_2, defined by

$$f_i = \left(\sum_{j=1}^{n} y_{ji}^2 \right)^{1/2} \tag{B.17}$$

each have a noncentral chi distribution with n degrees of freedom and the bivariate density, for the magnitude of ρ less than one, is given by [Miller, Bernstein, and Blumenson, 1958],[Krishnan, 1967]

$$p_{f_1f_2}(x_1,x_2) = A\ (\rho D_1D_2)^{1-m}\ x_1\ x_2\ \exp(-B_1x_1^2-B_2x_2^2)$$

$$\sum_{k=0}^{\infty} C_k\ I_r(2D_1x_1)\ I_r(2D_2x_2)\ I_r(2\rho B_1^{1/2}B_2^{1/2}x_1x_2) \tag{B.18}$$

where

TABLE B.1

SPECIAL CASES FOR PROBABILITY THAT ONE GENERALIZED RAYLEIGH VARIATE EXCEEDS ANOTHER WHEN THEY ARE INDEPENDENT

f_1	f_2	$(m_1,\beta_1;m_2,\beta_2)$	$\Pr\{f_2 > f_1\}$
Rician	Rayleigh	$(1,\beta_1;1,0)$	$[\Omega_2/(\Omega_1 + \Omega_2)]\ \exp[-\beta_1^2/(\Omega_1 + \Omega_2)]$
Nakagami	Nakagami	$(m_1,0;m_2,0)$	$I_t(m_1,m_2)$, $t = [m_1\Omega_2/(m_1\Omega_2+m_2\Omega_1)]$
Nakagami	Rayleigh	$(m_1,0;1,0)$	$[m_1\Omega_2/(\Omega_1+m_1\Omega_2)]^{m_1}$
Nakagami	*Gaussian	$(m_1,0;1/2,0)$	$I_t(m_1,1/2)$, $t = [2m_1\Omega_2/(\Omega_1+2m_1\Omega_2)]$
Rayleigh	Rician	$(1,0;1,\beta_2)$	$1 - [\Omega_1/(\Omega_1+\Omega_2)]\ \exp[-\beta_2^2/(\Omega_1+\Omega_2)]$
Rayleigh	Nakagami	$(1,0;m_2,0)$	$1 - [m_2\Omega_1(\Omega_2+m_2\Omega_1)]^{m_2}$
Rayleigh	Rayleigh	$(1,0;1,0)$	$[\Omega_2/(\Omega_1+\Omega_2)]$
Rayleigh	*Gaussian	$(1,0;1/2,0)$	$1 - [\Omega_1/(\Omega_1+2\Omega_2)]^{1/2}$
*Gaussian	Nakagami	$(1/2,0;m_2,0)$	$I_t(1/2,m_2)$, $t = [\Omega_2/(\Omega_2+2m_2\Omega_1)]$
*Gaussian	Rayleigh	$(1/2,0;1,0)$	$[\Omega_2/(\Omega_2+2\Omega_1)]^{1/2}$
*Gaussian	*Gaussian	$(1/2,0;1/2,0)$	$\frac{2}{\pi}\sin^{-1}[\Omega_2/(\Omega_1+\Omega_2)]^{1/2}$

*Gaussian refers to one-sided Gaussian.

$$r = k+m-1 \quad , \quad C_k = \frac{(2m-2)_k (m)_k}{(m-1)_k \; k!} \tag{B.19}$$

$$\omega_1 = \beta_1\Omega_2 - \rho\beta_2\Omega_1^{1/2}\Omega_2^{1/2} \quad , \quad \omega_2 = \beta_2\Omega_1 - \rho\beta_1\Omega_1^{1/2}\Omega_2^{1/2} \tag{B.20}$$

$$B_i = m[\Omega_i(1-\rho^2)]^{-1} \quad , \quad D_i = m\omega_i[\Omega_1\Omega_2(1-\rho^2)]^{-1} \tag{B.21}$$

$$A = 4B_1B_2(1-\rho^2)\Gamma(m)\ (m\Omega_1^{-1/2}\Omega_2^{-1/2})^{m-1}\ \exp[-(m\beta_1^2/\Omega_1) - (D_2^2/B_2)] \ . \tag{B.22}$$

Note that in the Rician case ($m = 1$), we have

$$\lim_{m\to 1} C_k = \lim_{m\to 1}\ (2m-2)_k \ / \ (m-1)_k \ = \ \varepsilon_k \tag{B.23}$$

where ε_k is Neumann's factor, which is defined to be equal to 2 when k is not zero, and equal to 1 when k is zero. As before, Eq.(B.18) is a valid bivariate probability density for all m not less than 1/2 and, in this context, is the joint probability density for correlated generalized Rayleigh variates with marginal distributions $f_1 \sim R(m,\beta_1,\Omega_1)$ and $f_2 \sim R(m,\beta_2,\Omega_2)$.

In general, the expression for either the joint moments or the correlation between f_1 and f_2 is very complicated [Krishnan, 1967]; however, for the Nakagami or Rayleigh cases, the normalized covariance of f_1^2 and f_2^2 (power correlation coefficient) is very simply related to the correlation coefficient of the underlying Gaussians, that is, [Nakagami, 1960]

$$\mathrm{Cov}(f_1^2, f_2^2) \;/\; [\mathrm{Var}(f_1^2)\ \mathrm{Var}(f_2^2)]^{1/2} \;=\; \rho^2 \;. \tag{B.24}$$

The probability that one generalized Rayleigh variate exceeds another in the correlated case is given by

$$\Pr\{f_2 > f_1\} = \int_0^\infty dx_2 \int_0^{x_2} dx_1 \;\; p_{f_1 f_2}(x_1, x_2) \;. \tag{B.25}$$

The inner integration can be evaluated by expanding the exponential and Bessel functions involving x_1 (Eq.(B.18)) into nested power series in the variable x_1 and interchanging the order of integration and summations. The outer integration is then readily evaluated by interchanging the order of integration and summations and using Eq.(A.27). Use of Eq.(A.28) and Eq.(A.11) then yields

$$\Pr\{f_2 > f_1\} = (1-\rho^2)^m \exp(-m\beta_1^2/\Omega_1) \sum_{k=0}^{\infty} (\Omega_2/\Omega_1)^{k+m} (-1)^k$$

$$G_k(m, D_1^2/B_1, D_2^2/B_2, \rho^2) \tag{B.26}$$

where

$$G_k(m,x,y,t) = \sum_{n=0}^{k} \frac{(2m-2)_n}{(m-1)_n \; n!} \; (txy)^{n/2} \sum_{s=0}^{k-n} \frac{(1-k-m)_s \, (1+k+m)_s}{(n+m)_s \quad s!}$$

$$t^s \;\; L_{k-n-s}^{(n+m-1)}(x) \; L_{k+m+s}^{(n+m-1)}(-y) \;. \tag{B.27}$$

As in the independent case, if the ratio (Ω_2/Ω_1) is greater than one, we can evaluate the complement of the probability (Eq.(B.12)).

For the case where f_1 and f_2 are independent (ρ=0), the only contribution from the sums over s and n in G_k is the first term (n=0,s=0) and Eq.(B.26) reduces to Eq.(B.11) for m_1 and m_2 equal to m. The result also simplifies considerably if f_1 and f_2 are both Nakagami m-distributed. In this case, D_1 and D_2 are both zero and the only contribution from the sum over n in G_k is the first term (n=0). By noting that

$$L_r^{(a)}(0) = \binom{r+a}{r} \quad , \tag{B.28}$$

we then have

$$\Pr\{f_2 > f_1\} = \binom{2m-1}{m} (1-\rho^2)^m \sum_{k=0}^{\infty} \frac{(2m)_k \; (m)_k}{k! \; (m+1)_k} (-1)^k (\Omega_2/\Omega_1)^{k+m}$$

$$_2F_1(-k,k+2m;m;\rho^2) \; . \tag{B.29}$$

If we further specialize to the case where f_1 and f_2 are both Rayleigh distributed (m=1), we obtain

$$\Pr\{f_2 > f_1\} = (1-\rho^2) \sum_{k=1}^{\infty} (-1)^{k+1} (\Omega_2/\Omega_1)^k \; _2F_1(1-k,k+1;1;\rho^2) \; . \tag{B.30}$$

APPENDIX C

SERIES EXPANSION FOR PROBABILITY OF ERROR

The probability of error is given by

$$\Pr\{E\} = \int_0^\infty dy \int_0^y dx \; p_{sz}(x,y) \; . \tag{C.1}$$

To develop a series expansion, the joint density of S and Z is expanded in a one-dimensional Taylor series about x equal to a, that is,

$$p_{sz}(x,y) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \; \frac{\partial^n}{\partial x^n} p_{sz}(x,y) \Big|_{x=a} \; . \tag{C.2}$$

Hence, the probability of error can be represented as

$$\Pr\{E\} = \int_0^\infty dy \int_0^y dx \; \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \; \frac{\partial^n}{\partial x^n} p_{sz}(x,y) \Big|_{x=a} \; . \tag{C.3}$$

By interchanging the order of integrations and summation and by noting that the partial derivatives evaluated at x equal to a are not a function of x, we obtain

$$\Pr\{E\} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty dy \; \frac{\partial^n}{\partial x^n} p_{sz}(x,y) \Big|_{x=a} \int_0^y dx \; (x-a)^n \; . \tag{C.4}$$

After evaluating the integration over x, we have

$$\Pr\{E\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_0^{\infty} dy \; \frac{\partial^n}{\partial x^n} p_{sz}(x,y)\Big|_{x=a} \; [(y-a)^{n+1} - (-a)^{n+1}]. \tag{C.5}$$

By using the binomial theorem (Eq.(A.9)) to expand $(y-a)^{n+1}$, Eq.(C.5) becomes

$$\Pr\{E\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_0^{\infty} dy \; \frac{\partial^n}{\partial x^n} p_{sz}(x,y)\Big|_{x=a} \; \sum_{m=1}^{n+1} \binom{n+1}{m} y^m (-a)^{n-m+1} \; . \tag{C.6}$$

After again interchanging the order of integration and summation and then interchanging the order of integration and partial differentiation, we obtain

$$\Pr\{E\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{m=1}^{n+1} \binom{n+1}{m} (-a)^{n-m+1} \frac{\partial^n}{\partial x^n} \int_0^{\infty} dy \; y^m \, p_{sz}(x,y)\Big|_{x=a} \; . \tag{C.7}$$

The joint density of S and Z equals the product of the conditional density of Z given S times the marginal density of S; Eq.(C.7) then yields

$$\Pr\{E\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{m=1}^{n+1} \binom{n+1}{m} (-a)^{n-m+1}$$

$$\frac{\partial^n}{\partial x^n} \; p_s(x) \int_0^{\infty} dy \; y^m \, p_{z|s}(y|S=x)\Big|_{x=a} \; . \tag{C.8}$$

Since the probability density of Z is symmetric about the origin, the remaining integral is half the m-th conditional absolute moment of Z given S, that is,

$$\langle |Z|^m \big| S=x \rangle = 2 \int_0^{\infty} dy \; y^m \, p_{z|s}(y|S=x) \tag{C.9}$$

where < > represents expectation. After translating the index of summation, we obtain the series expansion for Pr{E} about x equal to a:

$$\Pr\{E\} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^{n} \binom{n}{m} (-a)^{n-m} \frac{\partial^{n-1}}{\partial x^{n-1}} \; p_s(x) \; \langle |Z|^m \big| S=x \rangle \Big|_{x=a} . \tag{C.10}$$

For the case where a equals zero (Maclaurin series), the only contribution from the inner series is for m equal to n. In this case, we have [Hanlon and Gardner, 1979 and 1980]

$$\Pr\{E\} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^{m-1}}{\partial x^{m-1}} \; p_s(x) \; \langle |Z|^m \big| S=x \rangle \Big|_{x=0} \tag{C.11}$$

For the special case where S and Z are independent, the conditional absolute moments of Z are just the absolute moments of Z and Eq.(C.10) becomes

$$\Pr\{E\} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \; p_s^{(n-1)}(a) \sum_{m=1}^{n} \binom{n}{m} (-a)^{n-m} \langle |Z|^m \rangle . \tag{C.12}$$

Equation (C.11) becomes

$$\Pr\{E\} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} \; p_s^{(n-1)}(0) \; \langle |Z|^n \rangle . \tag{C.13}$$

It should be noted that in Eq.(C.12) the average power of the interference, i.e., $\langle Z^2 \rangle$, contributes to the second and higher terms of the series over n. In Eq.(C.13), the average power of the interference contributes only to the second term of the series. Since one would intuitively expect the probability of error to be strongly dependent on the average power of the interference, Eq.(C.13) would appear to be a more natural ordering likely to provide faster convergence. Additionally, the derivatives of the intended signal's

probability density, $p_s^{(n-1)}$, is in general of simpler form when evaluated for a equal to zero, as opposed to some other value of a. Similarly, Eq.(C.11) is generally preferable to Eq.(C.10).

As an example, the series expansion given by Eq.(C.13) is used to evaluate the probability of error when S and Z are independent, S is a generalized Rayleigh variate with distribution parameters (m_1, β_1, Ω_1) where $2m_1$ is restricted to be an integer, and $|Z|$ is a generalized Rayleigh variate with parameters (m_2, β_2, Ω_2). From Eq.(B.8), it is readily shown that the even derivatives of the density of S at the origin are zero and the odd derivatives at the origin are nonzero only if the order of the derivatives is greater than or equal to $(2m_1-1)$. The derivatives are then (for $n \geq m_1$)

$$p_s^{(2n-1)}(0) = (-1)^{n-m_1} [(2n)!/n!] (m_1/\Omega_1)^n \exp(-m_1\beta_1^2/\Omega_1) L_{n-m_1}^{(m_1-1)}(m_1\beta_1^2/\Omega_1) \quad \text{(C.14)}$$

The moments of $|Z|$ are given by Eq.(B.4) and the probability of error is readily shown to be [Hanlon and Gardner, 1980]

$$\Pr\{E\} = \frac{1}{2} \exp(-m_1\beta_1^2/\Omega_1) \sum_{n=0}^{\infty} (-1)^n (m_1\Omega_2/m_2\Omega_1)^{n+m_1}$$

$$L_n^{(m_1-1)}(m_1\beta_1^2/\Omega_1) \; L_{n+m_1}^{(m_2-1)}(-m_2\beta_2^2/\Omega_2) \; . \quad \text{(C.15)}$$

This result is consistent with Eq.(B.11).

APPENDIX D

CROSS-CORRELATION MOMENTS FOR RANDOM CODES

D.1. Moments for Integral Offset

The probability mass function for the code cross-correlation under the condition of integral offset for binary symmetric random codes of length L (L chips per data bit) is given by

$$\Pr\{c = k\} = \begin{cases} 2^{-L} \begin{pmatrix} L \\ \frac{L+k}{2} \end{pmatrix}, & \begin{matrix} -L \leq k \leq L \\ k \bmod 2 = L \bmod 2 \end{matrix} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{D.1}$$

It can be readily shown that the odd moments are zero and the even moments are given by

$$\langle c^{2n} \rangle = 2^{-L} \sum_{k=0}^{L} \begin{pmatrix} L \\ k \end{pmatrix} (2k - L)^{2n} = E_{2n}^{(-L)} \tag{D.2}$$

where $E_{2n}^{(-L)}$ is a generalized Euler number of order (-L). However, this form is not very convenient since L can be very large. By using the binomial theorem (Eq.(A.9)), Eq.(A.38), and the generating function for Stirling numbers of the first kind (Eq.(A.32)), the even moments can be expressed as a polynomial of degree n in the code length, i.e.,

$$\langle c^{2n}\rangle = \sum_{m=1}^{n} A_{m,n} L^{m} \qquad \text{for } n \geq 1 \tag{D.3}$$

where the coefficients ($A_{m,n}$) are given in terms of Stirling numbers of the first (S_1) and the second (S_2) kinds by the relation [Hanlon, Peterson, and Gardner, 1980]

$$A_{m,n} = \sum_{k=0}^{2n-m} 2^{k} \sum_{r=0}^{m-1} \binom{2n}{r} (-1)^{r} S_1(2n-r-k, m-r)\, S_2(2n-r, 2n-r-k) \ . \tag{D.4}$$

Provided n is not too large, this expression readily generates the polynomial coefficients. However, it is desirable to obtain closed-form expressions for the coefficients of at least the higher-order powers of L, since for large values of L (relative to n^2), the moments can be approximated very accurately by one or two higher-order terms of the polynomial. To obtain the closed-form expressions for the coefficients, we make use of the characteristic function for the cross-correlation. By using the probability mass function, we have

$$\langle e^{j\omega c}\rangle = 2^{-L} \sum_{k=0}^{L} \binom{L}{k} e^{j\omega(2k-L)} = (\cos\omega)^{L} \tag{D.5}$$

By applying the moment theorem and Taylor's theorem to Eq.(D.5), we obtain for the coefficients:

$$A_{m,n} = (-1)^{n} \frac{1}{m!} \frac{d^{2n}}{d\omega^{2n}} [\log(\cos\omega)]^{m} \Big|_{\omega=0} \ . \tag{D.6}$$

By applying the series expansion for log(cos ω) (Eq.(A.44)) and the result for powers of series (Eq.(A.29)), we have

$$A_{m,n} = (-1)^{m+n} \frac{(2n)!}{m!} H_{n-m,m} \tag{D.7}$$

where

$$H_{0,m} = 2^{-m} \ , \quad H_{k,m} = \frac{2}{k} \sum_{r=1}^{k} (rm-k+r)\, G_r\, H_{k-r,m} \ , \tag{D.8}$$

$$G_r = (-1)^r\, 2^{2r+1}\, (2^{2r+2}-1)\, B_{2r+2} \,/\, [(r+1)(2r+2)!] \ , \tag{D.9}$$

and the B_n are Bernoulli numbers. For example, for the highest-order coefficients we have

$$A_{n,n} = (2n)! \,/\, (n!\ 2^n) = (2n-1)!! \tag{D.10}$$

$$A_{n-1,n} = -n(n-1)\, A_{n,n} \,/\, 3 \tag{D.11}$$

$$A_{n-2,n} = -(n-2)(5n+1)\, A_{n-1,n} \,/\, 30 \ . \tag{D.12}$$

For large values of L relative to n^2, the even moments are approximately given by

$$\langle c^{2n} \rangle \simeq (2n-1)!!\, L^n\, [1 - n(n-1)/3L] \ . \tag{D.13}$$

This expression is exact for the second and fourth moments. Table D.1 lists the coefficients for the first five even moments.

TABLE D. 1

COEFFICIENTS FOR POLYNOMIAL EXPRESSIONS OF CROSS-CORRELATION MOMENTS FOR RANDOM CODES UNDER INTEGRAL OFFSET, $\langle c^{2n}\rangle = \sum_{m=1}^{n} A_{m,n} L^m$.

n	$A_{n,n}$	$A_{n-1,n}$	$A_{n-2,n}$	$A_{n-3,n}$	$A_{n-4,n}$
1	1				
2	3	-2			
3	15	-30	16		
4	105	-420	588	-272	
5	945	-6300	16380	-18960	7936

D.2. Moments for Arbitrary Offset

The moments for an integral offset can be used to generate the moments for an arbitrary offset. From Eq.(2.5), by taking the expectation over x, which is uniform on the interval (0,1), while holding c_0 and c_1 fixed, we obtain

$$\langle \hat{c}^n \big| c_0, c_1 \rangle = \frac{1}{n+1} \sum_{m=0}^{n} c_0^m \, c_1^{n-m} . \tag{D.14}$$

Since c_0 and c_1 are independent for random codes and the odd moments of c are zero, then the odd moments of $\hat{c}$ are zero and the even moments are given by

$$\langle \hat{c}^{2n} \rangle = \frac{1}{2n+1} \sum_{m=0}^{n} \langle c^{2m} \rangle \langle c^{2n-2m} \rangle$$

$$= \frac{1}{2n+1} \sum_{m=0}^{n} E_{2m}^{(-L)} E_{2n-2m}^{(-L)} . \tag{D.15}$$

It is clear from Eq.(D.3) and Eq.(D.15) that the even moments of $\hat{c}$ are also polynomials of degree n in the code length and can be expressed as

$$\langle \hat{c}^{2n} \rangle = \sum_{m=1}^{n} D_{m,n} L^m \qquad \text{for} \quad n \geq 1 . \tag{D.16}$$

By solving for the coefficients in terms of the $A_{m,n}$, we obtain [Hanlon, Peterson, and Gardner, 1980]

$$D_{m,n} = \frac{1}{2n+1} \sum_{k=0}^{n} \sum_{r=\max(0,k-n+m)}^{\min(m,k)} A_{r,k} A_{m-r,n-k} \tag{D.17}$$

where we define $A_{0,k} = \delta_{0k}$, that is, the Kronecker delta. For large values of L relative to n^2, we can approximate the even moments by two terms of the polynomial involving the highest-order powers of L, that is,

$$\langle \hat{c}^{2n} \rangle \simeq \frac{(2n-1)!!}{2n+1} L^n \sum_{m=0}^{n} \binom{n}{m} \binom{2n}{2m}^{-1} \{1 - [m^2+(n-m)^2-n]/3L\} . \tag{D.18}$$

This expression is exact for the second and the fourth moments. Table D.2 lists the coefficients for the first five even moments. It should be noted that the moments for arbitrary offsets differ significantly from those for integral offsets.

TABLE D.2

COEFFICIENTS FOR POLYNOMIAL EXPRESSIONS OF CROSS-CORRELATION MOMENTS FOR RANDOM CODES, $\langle \hat{c}^{2n} \rangle = \sum_{m=1}^{n} D_{m,n} L^m$.

n	$D_{n,n}$	$D_{n-1,n}$	$D_{n-2,n}$	$D_{n-3,n}$	$D_{n-4,n}$
1	2/3				
2	7/5	-4/5			
3	36/7	-64/7	32/7		
4	249/9	-912/9	1212/9	-544/9	
5	2190/11	-13680/11	34152/11	-38528/11	15872/11

VITA

Robert Charles Hanlon was born in Indianapolis, Indiana, on 10 December 1946. He received the Bachelor of Science degree in Engineering from Case Western Reserve University in 1968, and the Master of Science degree in Electrical Engineering from the University of Kansas in 1973. He was commissioned in the United States Air Force in February 1969. From 1969 to 1973, he served as a Communications-Operations Staff Officer at Headquarters, Air Force Communications Service, Richards-Gebaur AFB, Missouri. From 1973 to 1974, he was Chief, Communications Operations, 2129 Comm Sq, Ching Chuan Kang Air Base, Taiwan, Republic of China. From 1974 to 1975, he was Chief, Logistics Inspection, Tactical Communications Area, Langley AFB, Virginia. From 1975 to 1977, he was Chief, Communications Maintenance, 1913 Comm Sq, Langley AFB, Virginia. From 1977 to 1980, he was assigned to the Air Force Institute of Technology for doctoral study in communications engineering at the University of Illinois. He will receive the PhD degree in May 1980 and will be assigned to the Command and Control Technical Center, Headquarters, Defense Communications Agency. He currently holds the grade of Major in the Regular Air Force. He is a member of IEEE, Eta Kappa Nu, and the Air Force Association.

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1. Hanlon, R. C. and Gardner, C. S., "Error performance of direct sequence spread spectrum systems on non-selective fading channels," IEEE Trans. Comm., vol. COM-27, no. 11, pp. 1696-1700, November 1979.

2. Hanlon, R. C. and Gardner, C. S., "Error performance of direct sequence spread spectrum systems on nonselective generalized Rayleigh fading channels," submitted to IEEE Trans. Comm., February 1980.

3. Hanlon, R. C., Peterson, G. D., and Gardner, C. S., "An empirical estimate for the cross-correlation moments for Gold codes," submitted to IEEE Trans. Comm., February 1980.